EQUIVALENCE OF CONNECTIVITY MAPS AND PERIPHERALLY CONTINUOUS TRANSFORMATIONS

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In [1] and [2] O. H. Hamilton and J. Stallings have shown that a local connectivity mapping, and hence a connectivity mapping, of a locally peripherally connected polyhedron into a regular Hausdorff space is peripherally continuous. The purpose of this paper is to prove the converse of this theorem.

Some definitions will now be recalled. A mapping $f: S \to T$ is a connectivity mapping if for every connected set A in S, the set $g(A)$ is connected, where $g: S \to S \times T$ is the graph map of f defined by $g(p) = (p, f(p))$ [1, p. 750]. The mapping f is a local connectivity mapping if there is an open covering $\{U_\alpha\}$ of S such that $f|U_\alpha$ is a connectivity mapping for every α [2, p. 249]. The mapping f is peripherally continuous if for every point p in S and for every pair of open sets U and V containing p and $f(p)$, respectively, there is an open set $N \subset U$ and containing p such that $f(F(N)) \subset V$, where $F(N)$ is the boundary of N [1, p. 751]. A space S is locally peripherally connected if every point has arbitrarily small neighborhoods with connected boundary [2, p. 252].

In this paper S will denote a connected, locally connected, locally peripherally connected, unicoherent metric space and T a space such that $S \times T$ is completely normal.

The following lemma, proved by Stallings [2, p. 255], is used in the proof of Theorem 1.

LEMMA 1. *If $f: S \to T$ is peripherally continuous, then for every point p in S and every pair of open sets U and V containing p and $(p, f(p))$, respectively, there is an open connected set $N \subset U$ and containing p such that $F(N)$ is connected and $g(F(N)) \subset V$.*

LEMMA 2. *Let W be an open connected subset of S such that $F(W)$ is connected. Let W_1 and W_2 be open connected sets such that $W_1 \cap W_2 \neq \emptyset$, $F(W_1)$ and $F(W_2)$ are connected, and* $\mathrm{cl}(W_1) \cup \mathrm{cl}(W_2) \subset W$. *Then there is a connected open set W_3 such that* (1) $W_1 \cup W_2 \subset W_3 \subset W$, (2) *$F(W_3)$ is contained in $F(W_1) \cup F(W_2)$, and* (3) *$F(W_3)$ is connected.*

PROOF. The proof is similar to the proof of Lemma 1. Let $X = W_1 \cup W_2$. Then $F(X)$ is connected and separates $F(W)$ and X. Let

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$C=F(X)\cup\{y\in W;\ F(X)$ separates y and $F(W)\}$ and $W_3=$component of int C containing X. Then by standard theorems concerning unicoherence [**3**, p. 51], $F(W_3)\subset F(X)$ and $F(W_3)$ is connected.

The following theorem is the converse of Hamilton's and Stallings' theorem.

THEOREM 1. *If $f: S\rightarrow T$ is peripherally continuous, then f is a connectivity map.*

PROOF. Suppose that f is not a connectivity map and let A be a connected subset of S such that $g(A)=M\cup N$, where M and N are separated. Let $g^{-1}(M)=H$ and $g^{-1}(N)=K$. Then $A=H\cup K$, where $H\cap K=\emptyset$. Since A is connected H and K are not separated and hence one must contain a limit point of the other. Let p be a point of H that is a limit point of K. Since $S\times T$ is completely normal there exist open disjoint sets U and V in $S\times T$ containing M and N, respectively.

Let R be an open set containing p such that A is not contained entirely in R. By Lemma 1 there is an open connected set W containing p and contained in R such that W and $F(W)$ are both connected and $g(F(W))\subset U$. Since p is a limit point of K there is a point q of K in W.

Let Q be the collection of all open connected sets D such that q is in D, $\text{cl}(D)\subset W$, $F(D)$ is connected, and $g(F(D))\subset V$. The collection Q is nonempty since f is peripherally continuous at the point q. Denote by Q^* the point-set union of all sets in Q. Then Q^* is an open subset of W. Since the connected set A intersects both Q^* and $S-Q^*$, it follows that $A\cap F(Q^*)\neq\emptyset$.

Since $F(Q^*)\cap A\neq\emptyset$, then $F(Q^*)$ either contains a point of H or a point of K. Suppose there is a point h in $F(Q^*)\cap H$. Then there is an open set E containing h but not q such that $F(E)$ is connected and $g(F(E))\subset U$. Since h is a limit point of Q^*, E must intersect some set D belonging to the collection Q. Now $E\not\subset D$ since h is in $E-D$ and $D\not\subset E$ since q is in $D-E$. Thus E and D both have points interior and exterior to one another and $F(D)$ and $F(E)$ being connected implies $F(D)\cap F(E)\neq\emptyset$. But this contradicts the fact that $g(F(D))\subset V$, $g(F(E))\subset U$ and $U\cap V=\emptyset$. Hence $F(Q^*)\cap H=\emptyset$ and therefore $F(Q^*)\cap K\neq\emptyset$.

Let k be a point of $F(Q^*)\cap K$. Now k is not a point of $F(W)$ since $g(F(W))\subset U$ and $g(k)$ is in V. Thus k is in W and there is an open connected set W_1 containing k and contained in W such that $F(W_1)$ is connected, $\text{cl}(W_1)\subset W$ and $g(F(W_1))\subset V$. Since k is a limit point of Q^* there is a set W_2 in the collection Q such that $W_1\cap W_2\neq\emptyset$.

Now form the set W_3 referred to in Lemma 2. By this lemma the set W_3 is open, connected, $F(W_3)$ is connected, $\mathrm{cl}(W_3) \subset W$, and q is in W_3. Further, $g(F(W_3)) \subset V$ since $F(W_3) \subset F(W_1) \cup F(W_2)$. Therefore W_3 possesses all the requirements to belong to Q, but W_3 is not in Q since k is in $(W_3 \cap F(Q^*))$. Therefore the assumption that $g(A)$ is not connected leads to a contradiction. Hence f is a connectivity map.

Stallings' theorem, [2, p. 253], and Theorem 1 imply, in particular, that on an n-cell, $n \geqq 2$, into itself there is no distinction among local connectivity maps, connectivity maps, and peripherally continuous transformations. Thus, the question posed on p. 752 of [1] and question 5, p. 262 of [2] are answered. The following theorem will complete the theory of equivalence of the local connectivity maps and the connectivity maps of an n-cell, $n = 1, 2, \cdots$, into itself.

THEOREM 2. *If f is a local connectivity map of the closed unit interval I into itself, then f is a connectivity map.*

PROOF. Since f is a local connectivity map there is an open covering $\{U_\alpha\}$ of I such that f restricted to U_α is a connectivity map for each α. Since I is compact the covering $\{U_\alpha\}$ can be reduced to an irreducible number of intervals $I_1, \cdots, I_n$, such that $I_i \cap I_{i+1} \neq \emptyset$, and f is a connectivity map on each I_i. Then if K is any connected subset of I, K is an interval and $K = (K \cap I_1) \cup \cdots \cup (K \cap I_n)$, where each $K \cap I_i$ is an interval contained in I_i. Thus $g(K \cap I_i)$ is connected and since $g(K \cap I_i) \cap g(K \cap I_{i+1}) \neq \emptyset$, $g(K)$ is connected. Therefore f is a connectivity map.

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