PARTIALLY ORDERED GROUPS OF THE SECOND AND THIRD KINDS

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1. Introduction. Let $G$ be both a group and a partially ordered set. An element $a$ of $G$ is called a left [right] conserver if

$$x \leq y (x, y \in G) \Rightarrow ax \leq ay [xa \leq ya]$$

and a left [right] inverter if

$$x \leq y (x, y \in G) \Rightarrow ax \geq ay [xa \geq ya].$$

We shall call an element of $G$ a conserver [inverter] if it is both a left and a right conserver [inverter].

If every element of $G$ is a conserver, then $G$ is a partially ordered group (abbreviated “po-group”) in the usual sense; we shall also say that $G$ is a po-group of the first kind. If every element of $G$ is a conserver or an inverter, and not every element of $G$ is a conserver, then we shall call $G$ a po-group of the second kind. A familiar example is the multiplicative group of all nonzero real numbers with the usual ordering. The stipulation that not every element of $G$ is a conserver excludes the possibility that $G$ be trivially ordered, and it is then clear that no element of $G$ can be both a conserver and an inverter.

The structure of totally ordered groups (“o-groups”) of the second kind has been reduced to that of o-groups of the first kind by J. A. H. Shepperd [1]. (What he calls a “betweenness group” is either an o-group of the first or second kinds, or a finite group of order 4.) The first main result of the present note (Theorem 1) is an extension of Shepperd’s result from o-groups to po-groups. The proof has also been simplified by avoiding reference to the betweenness relation.

Totally ordered semigroups (“o-semigroups”) of the second kind have been considered by the author [2] in the commutative case, and by J. Gilder [3] and K. Keimel [4] in general. Following Gilder’s terminology, we define a po-group of the third kind to be a group $G$ endowed with a nontrivial partial order, such that each element of $G$ is either a left conserver or a left inverter, and also either a right conserver or a right inverter, and such that $G$ contains an element which conserves on one side and inverts on the other. Theorem 2 gives a reduction of these to po-groups of the first kind.
Added in proof. The author regrets that he was not aware at the time of writing this paper that a result equivalent to Theorem 1 below had been obtained by J. F. Andrus and A. T. Butson [5] for a connected po-group of the second kind. The two approaches are also different, however, that the equivalence of the results is not apparent. Their subgroup $S_0$ is the (directed) po-subgroup of my subgroup $H$ generated by its positive cone $H_+$. My subset $I_-$ of $G \setminus H$ is the union of those cosets of $S_0$ in $G$ which belong to their subset $T_0$ of the factor group $G/S_0$. Removing their requirement of connectedness actually simplifies more than it complicates. Thus the six properties (i)–(vi) which $T_0$ must have in their Theorem 5 reduce to (i), (iv), (v), and the requirement that there exists a subgroup $H$ of index 2 in $P$ ($=G/S_0$) such that $T_0 \subseteq G \setminus H$. (Incidently, (v) should read “$a + T_0 = T_0$.”) This is an immediate consequence of Theorem 1 below, in the case $H_+ = 0$.

2. Partially ordered groups of the second kind. We denote the identity element of $G$ by $e$, and set

$$G_+ = \{ a \in G : a \geq e \}, \quad G_- = \{ a \in G : a \leq e \}.$$  

For any subset $A$ of $G$, we let $A_+ = A \cap G_+$, $A_- = A \cap G_-$, and $A^{-1} = \{ a^{-1} : a \in A \}$. A (possibly empty) subset $A$ of $G$ is called an upper [lower] class in $G$ if $a \in A$, $x \in G$, and $a < x$ [$a > x$] imply $x \in A$. The empty set is denoted by $\emptyset$, and $A \setminus B$ means the set of elements of $A$ not in $B$.

By the order dual $G^*$ of $G$ we mean the group $G$ endowed with the dual $^* \colon a \mapsto a^*$. $G$ and $G^*$ have the same sets of left [right] conservers and inverters.

Theorem 1. Let $G$ be a po-group of the second kind. Let $H[I]$ be the set of conservers [inverters] of $G$. Then $H$ is a subgroup of $G$ of index 2, and $I$ is its other coset. $H$ is a po-group of the first kind, and $H_+$ is normal in $G$. $H$ and $I$ are convex subsets of $G$, and, by passing to the order dual of $G$ if necessary, we can assume that $H$ is an upper class and $I$ a lower class in $G$. In particular, $H_+ = G_+$, $I_+ = \emptyset$, and $I_-$ is a lower class in $G$. The set $I_-$ has the following properties:

(N1) $I_-$ is normal in $G$.
(N2) $I_+^{-1} = I_-$.
(N3) $I_-$ contains $H_+ I_-$, $I_- H_+$, $H_+ I_-$, and $I_- H_-$.

The order relation $\leq$ can be described in terms of $H_+$ and $I_-$ as follows:

(O1) If $x \in H$, $y \in H$, then $x \leq y \iff x^{-1} y (\text{or } y x^{-1}) \in H_+$.
(O2) If $x \in I$, $y \in I$, then $x \leq y \iff y x^{-1} (\text{or } y^{-1} x) \in H_+$.
(O3) If $x \in I$, $y \in H$, then $x \leq y \iff x y^{-1} (\text{or } y^{-1} x \text{ or } x^{-1} y \text{ or } y x^{-1}) \in I_-.$
If \( x \in H, y \in I \), then \( x \leq y \) never holds.

If \( H \) is directed, then \( I_- \) must be \( I \) or \( \emptyset \). If \( I_- = I \), then every element of \( I \) is less than every element of \( H \). If \( I_- = \emptyset \), then no element of \( I \) is comparable with any element of \( H \).

Conversely, let \( G \) be a group containing a subgroup \( H \) of index 2, and let \( I = G \setminus H \). Assume that \( H \) is a po-group of the first kind, and that its positive cone \( H^+ \) is normal in \( G \). Let \( I_- \) be a subset of \( I \) having properties (N1–3). Define \( \leq \) in \( G \) by (O1–4). This agrees with the given partial order in \( H \) by (O1), and \( G \) becomes thereby a po-group of the second kind such that \( H[I] \) is the set of conservers [inverters] of \( G \).

Remarks. (1) If (N1) and (N2) hold, and if \( I_- \) contains any one of the four product sets in (N3), then it contains the other three.

(2) Regarding the parenthetical assertions in (O3), we note that if \( I_- \) is any subset of \( I \) satisfying (N1) and (N2), and if any one of the four products \( xy^{-1}, y^{-1}x, x^{-1}y, yx^{-1} \) belong to \( I_- \), then so do the remaining three. A similar remark applies to (O1) and (O2), since \( H^+ \) is normal in \( G \).

Proof. Evidently the product of two conservers or of two inverters is a conserver, while that of a conserver and an inverter is an inverter. Since the identity element \( e \) of \( G \) is a conserver, the inverse of a conserver [inverter] must be of the same type. From these remarks it is clear that \( H \) is a subgroup of \( G \) of index 2, that \( I = G \setminus H \), and that \( H \) is a po-group of the first kind.

If \( p \in H^+ \) and \( u \in I \), then from \( e \leq p \) we have \( u \leq pu \) and \( e = u^{-1}u \leq u^{-1}pu \). Thus \( u^{-1}H^+u \subseteq H^+ \). Since \( H^+ \) is normal in \( H \), this shows that it is normal in \( G \).

To show that \( H \) is convex in \( G \), it clearly suffices to show that \( e < u < h (h \in H, u \in I) \) is impossible. Multiplying \( e < u < h \) on the left by the inverter \( u \), and on the right by the conserver \( h \), we obtain \( u > u^2 > uh \) and \( h < uh < h^2 \). But this yields \( u > uh > h \), contrary to \( u < h \).

To show that \( I \) is convex in \( G \), suppose that \( u > h > u' (h \in H; u, u' \in I) \). Then \( e < hu^{-1} < u'u^{-1} \). Since \( u'u^{-1} \in H \) and \( huu^{-1} \in I \), this contradicts the convexity of \( H \).

From \( G = H \cup I \) it follows that \( H \) must be either an upper class or a lower class in \( G \). By passing to the order dual of \( G \), if necessary, we can assume that \( H \) is an upper class. Then \( I \) is a lower class. Since \( e \in H \), we have \( G^+ \subseteq H \), and hence \( H^+ = G^+ \) and \( I^+ = \emptyset \). \( I_- \) is clearly a lower class in \( I \), hence also in \( G \).

If \( u \in I_- \) and \( v \in I \), then from \( u < e \) we have \( uv > v \) and \( v^{-1}uv < v^{-1}v = e \), hence \( v^{-1}uv \in I_- \). Similarly, \( h^{-1}uh \in I_- \) for every \( h \) in \( H \), which proves (N1). To show (N2), we note that \( u < e \) (\( u \in I \)) implies \( e = uu^{-1} > eu^{-1} = u^{-1} \), hence \( u^{-1} \in I_- \). By Remark (1), to estab-
lish (N3) we need only show that $H+I_\subseteq I_-$. From $h>e$, $u<e$ ($h\in H$, $u\in I$), we have $hu<eu=e$, so that $hu\in I_-$. (O1) is a standard fact about po-groups, and (O4) is just the assertion that $H$ is an upper class in $G$. To show (O2), we note that $x\leq y$ is equivalent to $xy^{-1}\geq e$, since $y\in I$. To show (O3), we observe that $x\leq y$ is now equivalent to $xy^{-1}\leq yy^{-1}=e$, since $y\in H$.

If $H$ is directed, then $H=H_+H_-$, and (N3) implies that $HI_-\subseteq I_-$. If $I_\neq \emptyset$, let $u\in I_-$. Then $I=H_\subseteq I_-$, whence $I_-\subseteq I$.

Turning to the converse, let $H$ be a subgroup of $G$ of index 2, and let $H$ be a po-group of the first kind such that $H_+$ is normal in $G$. Let $I=G\setminus H$. Let $I_-$ be a subset of $I$ having properties (N1-3), and define $\leq$ in $G$ by (01-4). (O1) asserts that the restriction of $\leq$ to $H$ shall coincide with the given partial ordering of $H$.

That the relation $\leq$ is reflexive and antisymmetric is clear. To prove that it is transitive, let $x\leq y$ and $y\leq z$ ($x, y, z\in G$). (O4) implies that if $x\in H$ then $y\in H$, and if $y\in H$ then $z\in H$. Since we do not need to consider the case $x, y, z\in H$, we are left with three cases.

**Case** $x\in I$, $y\in H$, $z\in H$. By (O3) and (O1) we have $x^{-1}y\in I_-$ and $y^{-1}z\in H_+$. By (N3), $x^{-1}z=(x^{-1}y)(y^{-1}z)\in I_-$, and $x\leq z$ by (O3).

**Case** $x\in I$, $y\in I$, $z\in H$. By (O2) and (O3) we have $xy^{-1}\in H_+$ and $yz^{-1}\in I_-$. By (N3), $xz^{-1}=(xy^{-1})(yz^{-1})\in I_-$, and $x\leq z$ by (O3).

**Case** $x\in I$, $y\in I$, $z\in I$. By (O2) we have $xy^{-1}\in H_+$ and $yz^{-1}\in H_+$. Hence $xz^{-1}=(xy^{-1})(yz^{-1})\in H_+$, and $x\leq z$ by (O2).

Hence $G$ is a po-set under $\leq$. All that remains is to show that every element of $H[I]$ is a conserver [inverter]. Since every element of $H$ is the product of two elements of $I$, it suffices to show that every element of $I$ is an inverter.

Let $u\in I$, and let $x\leq y$. The case $x\in H$, $y\in I$ is excluded by (O4), and we consider the remaining three.

**Case** $x\in H$, $y\in H$. By (O1), $x^{-1}y$ and $yx^{-1}\in H_+$. Hence $(ux)^{-1}(uy)$ and $(yu)(xu)^{-1}\in H_+$, and we infer from (O2) that $uy\leq ux$ and $yu\leq xu$.

**Case** $x\in I$, $y\in I$. By (O2), $xy^{-1}$ and $y^{-1}x\in H_+$. Hence $(xu)(yu)^{-1}$ and $(yu)^{-1}(ux)\in H_+$, and we infer from (O1) that $yu\leq xu$ and $uy\leq ux$.

**Case** $x\in I$, $y\in H$. By (O3), $x^{-1}y$ and $yx^{-1}\in I_-$. Hence $(ux)^{-1}(uy)$ and $(yu)(xu)^{-1}\in I_-$, and we infer from (O3) that $uy\leq ux$ and $yu\leq xu$.

This concludes the proof of the theorem.

Let us consider all possible ways of extending a given po-group $H$ of the first kind to a po-group $G$ of the second kind, such that $H$ is the set of conservers of $G$. In the first place, $G$ must be an extension of $H$ by the cyclic group $C_2$ of order 2; the Schreier theory tells us how to find all such. Call $G$ "suitable" if $H_+$ is normal in $G$; there is at
least one suitable $G$, namely the direct product $H \times C_2$. Any suitable $G$ can be partially ordered in the desired fashion by choosing $I_-$ so as to satisfy (N1-3). This can always be done by choosing $I$ or $\emptyset$ for $I_-$, and these are the only possibilities if $H$ is directed. If $G$ itself is to be directed, only $I_- = I$ is possible, and then every element of $H$ exceeds every element of $I$. In this case we note that $G$ will be lattice-ordered or totally ordered if and only if the same holds for $H$.

If $H$ is trivially ordered, then $I_- \neq \emptyset$, since $G$ cannot be trivially ordered. As a simple example with $I_- \neq I$, let $G$ be the infinite cyclic group generated by $a$, let $H$ be the subgroup generated by $a^2$, and let $I_- = \{a, a^{-1}\}$. The resulting partial order on $G$ has a saw-tooth nature:

$$\cdots > a^{-3} < a^{-2} > a^{-1} < e > a < a^2 > a^3 < \cdots .$$

3. Partially ordered groups of the third kind. We define the following four subsets of a po-group $G$ of the third kind. Let $C_2 = \{0, 1\}$ be the additive group of integers mod 2, so that $1 + 1 = 0$. For $i$ and $j$ in $C_2$ let $G_{ij}$ be the set of all elements $a$ of $G$ such that $a$ is a left conservator if $i = 0$, a left inverter if $i = 1$, a right conservator if $j = 0$, and a right inverter if $j = 1$.

From the way left and right conservers and inverters multiply,

$$G_{ij}G_{kl} = G_{i+k,j+l} \quad (i, j, k, l \in C_2).$$

By definition of po-group of the third kind,

$$G = G_{00} \cup G_{01} \cup G_{10} \cup G_{11}, \quad G_{01} \cup G_{10} \neq \emptyset.$$
po-group of the first kind such that its positive part $P$ satisfies (N'1–2), with $G_{ij}$ replaced by $H_{ij}$. Define a relation $\leq$ on $G$ by (O'1–2), similarly modified, with $\leq$ never holding between elements of distinct $H_{ij}$. Then $G$ becomes a po-group of the third kind, with $G_{ij} = H_{ij}$. The same holds in the event $G/H_{00} \cong C_2$ if we let $H_{11} = \emptyset$, and either $H_{01} = \emptyset$ or $H_{10} = \emptyset$.

**Proof.** The first three sentences are obvious. (N'1) then follows from Theorem 1. To show (N'2), let $p \in P$ and let $a \in G_{01}$. From $e < p$ we have $a < ap$, since $a$ is a left conserver, and hence $e > apa^{-1}$, since $a$ is a right inverter. Thus $apa^{-1} \in P^{-1}$. The proof for $a$ in $G_{10}$ is similar.

We note that the identity element $e$ of $G$ cannot be comparable with any element of $G_{01}$ or $G_{10}$. For if $e < a$ ($a \in G_{01}$), then $a < a^2$ since $a$ is a left conserver, and hence $e > a^2$, since $a$ is a right inverter. The argument is similar if $e > a$, or if $a \in G_{10}$. Moreover, $e$ cannot be comparable with an element $a$ of $G_{11}$. For suppose $a \in G_{01}$. Then $ba < b$ and $bab^{-1} > e$, since $b^{-1} \in G_{01}$. But $bab^{-1} \in G_{01}G_{11}G_{01} = G_{11}$, and $a < e < bab^{-1}$ would violate the convexity of $I = G_{11}$ in the po-group $G_{00} \cup G_{11}$ of the second kind (Theorem 1). The argument is similar if $e < a$.

Now let $a$ and $b$ be any two elements of $G$ such that $a < b$. Then $aa^{-1} < ba^{-1}$ or $aa^{-1} > ba^{-1}$, depending on whether $a^{-1}$ is a right conserver or a right inverter. In either case, $ba^{-1}$ is comparable with $e$, and so belongs to $G_{00}$. Hence $a$ and $b$ belong to the same coset $G_{ij}$.

To show (O'1), $x \leq y \iff e \leq x^{-1}y \iff x^{-1}y \in P$, since $x$ is a left conserver. As for (O'2), $x$ is a left inverter, and so $x \leq y \iff e \geq x^{-1}y \iff x^{-1}y \in P^{-1}$.

Proceeding to the converse, let us introduce the notation $P_0 = P$, $P_1 = P^{-1}$, where $0, 1 \in C_2$. Then, for any $k$ in $C_2$, $P_k = P_{k+1}$. The modified rules (N'1–2) and (O'1–2) can then be condensed into single formulae:

$(N')$ if $a \in H_{ij}$, then $aPa^{-1} \subseteq P_{i+j}$;

$(O')$ if $x, y \in H_{kl}$, then $x \leq y \iff x^{-1}y \in P_k$.

It is evident that $\leq$ defined by (O') is reflexive and symmetric. If $x \leq y$ and $y \leq z$, then $x, y,$ and $z$ all belong to the same $H_{kl}$, and $x^{-1}z = (x^{-1}y)(y^{-1}z) \in P_kP_k \subseteq P_k$, whence $\leq$ is transitive.

To show that $H_{ij} = G_{ij}$, let $a \in H_{ij}$ and let $x \leq y$. Then $x$ and $y$ belong to the same $H_{kl}$ and $x^{-1}y \in P_k$. From $(ax)^{-1}(ay) = x^{-1}y \in P_k$, and $ax, ay \in H_{i+k,j+l}$, (O') gives $ax \leq ay$ if $i = 0$ and $ay \leq ax$ if $i = 1$. Since this is independent of $k$ and $l$, we conclude that $a$ is a left conserver if $i = 0$ and a left inverter if $i = 1$.

From $(xa)^{-1}(ya) = a^{-1}(x^{-1}y)a \in P_{k+k(i+j)} = P_{(i+k)+j}$ by (N'), and $xa, ya \in H_{i+k,j+l}$, we conclude from (O') that $xa \leq ya$ if $j = 0$ and $ya \leq xa$ if $j = 1$. Hence $a$ is a right conserver if $j = 0$ and a right inverter if $j = 1$. 
Let $H$ be a po-group of the first kind. We saw at the conclusion of §2 that $H$ can be extended in at least one way to a po-group $G$ of the second kind. This is not so if $G$ is to be of the third kind. In fact it is possible if and only if there exists an automorphism of $H$ the square of which is inner, and which maps the positive cone $P$ of $H$ into $P^{-1}$. This is always possible if $H$ is abelian, since $x\rightarrow x^{-1}$ is then an automorphism with these properties. But it is impossible if $H$ is a group every automorphism of which is inner. For example, let $H$ be the group of rational matrices of the form

$$
\begin{pmatrix}
a & b \\
0 & 1
\end{pmatrix}
$$

with $a > 0$, and define $P(H)$ to be the set of all such matrices with $a \geq 1$.

References


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