A SCHWARZ LEMMA FOR BOUNDED SYMMETRIC DOMAINS

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The purpose of this note is to generalize Pick's invariant formulation of the classical Schwarz lemma. For the four classical types of bounded symmetric domains such a generalization was given by K. H. Look [3]; the present treatment will be independent of classification theory and will also include the exceptional Cartan domains. The results are also independent of the particular realization of the domain in $C^n$, they depend only on its structure as a hermitian manifold. The results will therefore be formulated for hermitian symmetric spaces of noncompact type; these are known to be in one-to-one correspondence with the holomorphic equivalence classes of bounded symmetric domains. We shall make use of Harish-Chandra's canonical realization of the hermitian symmetric spaces as bounded domains; this could perhaps be avoided, but it makes the proofs considerably simpler.

In the following $M = G/K$ will be a hermitian symmetric space of noncompact type; the identity coset will be denoted by $p_0$, $\mathfrak{g}$ and $\mathfrak{f}$ will denote the Lie algebras of $G$ and $K$, respectively, and $H_a$, $E_a$, $\cdots$ will be a Weyl basis of $\mathfrak{g}$ with respect to a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{f}$. By a result of Harish-Chandra there exists a set $\Delta$ of strongly orthogonal roots of $\mathfrak{g}$ such that $a = \sum_{a \in \Delta} R(E_a + E_{-a})$ is a Cartan subalgebra of the symmetric pair $(\mathfrak{g}, \mathfrak{f})$. So every point $p \in M$ can be represented in the form $p = k \exp(\sum_{a \in \Delta} t_a (E_a + E_{-a})) \cdot p_0$ with $k \in K$, $t_a \geq 0$.

For any $p, q \in M$ we denote by $d(p, q)$ the distance of $p$ and $q$ in the metric induced by the hermitian structure of $M$. In any realization of $M$ as a complex domain this is the Bergman metric. We denote by $d^*(p, q)$ the Carathéodory distance, which is defined by

$$d^*(p, q) = \sup_{f \in \mathcal{F}} d_U(f(p), f(q)),$$

where $\mathcal{F}$ is the family of all holomorphic maps of $M$ into the unit disc $U \subset \mathbb{C}$, and $d_U$ is the Poincaré-Bergman distance function on $U$.

**Lemma.** Let $p = \exp(\sum_{a \in \Delta} t_a (E_a + E_{-a})) \cdot p_0$, $t_a \geq 0$ ($a \in \Delta$). Then
Proof. The first statement follows from the known fact that the orbits of \( p_0 \) under one-parameter groups generated by elements of \( \alpha \) are geodesics; thus

\[
\exp \left( s \left( \sum_{\alpha \in \Delta} t_\alpha \right)^{-1/2} \sum_{\alpha \in \Delta} t_\alpha (E_\alpha + E_{-\alpha}) \right) \cdot p_0 \left( 0 \leq s \leq \left( \sum_{\alpha \in \Delta} t_\alpha^2 \right)^{1/2} \right)
\]

is a geodesic segment in arc-length parameters connecting \( p_0 \) and \( p \).

For the second statement we use the Harish-Chandra realization of \( M \) as a bounded domain. We denote by \( \Phi \) the set of positive roots of \( g \) which are not roots of \( f \), and by \( \mathfrak{p}^- \) the complex subspace of \( g^c \) spanned by the vectors \( E_{-\alpha} (\alpha \in \Phi) \). The Harish-Chandra realization \( \eta: M \to \mathfrak{p}^- \) is given for any \( p = k \exp(\sum_{\alpha \in \Delta} t_\alpha (E_\alpha + E_{-\alpha})) \cdot p_0 \) by \( \eta(p) = \text{ad}(k) \sum_{\alpha \in \Delta} r_\alpha E_{-\alpha} \), where \( r_\alpha = \tanh t_\alpha (\alpha \in \Delta) \). We denote the domain \( \eta(M) \subset \mathfrak{p}^- \) by \( D \).

Now let \( p \) be as in the statement of the Lemma, and let \( \alpha_0 \in \Delta \) be such that \( t_{\alpha_0} = \text{Max}_{\alpha \in \Delta} t_\alpha \). We write \( r_{\alpha_0} = \tanh t_{\alpha_0} \). Let \( f: M \to U \) be a holomorphic function such that \( f(p_0) = 0 \). Defining the function

\[
\phi : U \to U \text{ by } \phi(z) = f(\eta^{-1}(z r_{\alpha_0}^{-1} \eta(p)))
\]

we have, by the classical Schwarz lemma, \( |\phi(z)| \leq |z| \) for all \( z \in U \). In particular, for \( z = r_{\alpha_0} \), it follows that \( |f(p)| \leq r_{\alpha_0} \). In the definition of the Carathéodory distance it is sufficient to consider functions \( f \in F \) such that \( f(p_0) = 0 \), since \( U \) is homogeneous. Hence, from what we just proved it follows that

\[
d^*(p_0, p) \leq d_U(0, r_{\alpha_0}) = t_{\alpha_0}.
\]

On the other hand, let \( g : D \to U \) be defined by \( g(\sum_{\alpha \in \Phi} z_\alpha E_{-\alpha}) = z_{\alpha_0} \), and let \( f_1 = g \circ \eta \). Then \( f_1 \in F \), and

\[
d^*(p_0, p) \geq d_U(f_1(p_0), f_1(p)) = d_U(0, r_{\alpha_0}) = t_{\alpha_0},
\]

finishing the proof of the Lemma.

Proposition 1. Let \( M \) be a hermitian symmetric space of rank \( l \) and let \( f: M \to M \) be a holomorphic function. Then, for any \( p, q \in M \),

\[
d(f(p), f(q)) \leq l^{1/2} d(p, q).
\]

The constant \( l^{1/2} \) is the best possible.

Proof. Given any pair of points \( p_1, p_2 \in M \) we can find an element \( g \) in \( G \) such that \( gp_1 = p_0, \ g p_2 = \exp(\sum_{\alpha \in \Delta} t_\alpha (E_\alpha + E_{-\alpha})) \cdot p_0 \). By the Lemma it follows that

\[
d(f(p), f(q)) \leq l^{1/2} d^*(f(p), f(q)) \leq l^{1/2} d^*(p, q) \leq l^{1/2} d(p, q)
\]
proving the first statement.

To see that $l^{1/2}$ is best possible, let $p = \exp t(E_{a_0} + E_{-a_0}) \cdot p_0$ with some $t > 0$ and $a_0 \in \Delta$. Define $g : D \to D$ by $g(\sum_{a \in \Phi} z_a E_{-a}) = z_{a_0} \sum_{a \in \Delta} E_{-a}$, and let $f = \eta^{-1} \circ g \circ \eta$. By Lemma 1 we have $d(p_0, p) = t$ and $d(f(p_0), f(p)) = l^{1/2} t$, finishing the proof.

Remark. The Proposition remains true, by the same proof, for holomorphic functions $f : M_1 \to M_2$ where $M_1$, $M_2$ are hermitian symmetric spaces and $l$ is the rank of $M_2$.

Next we give an infinitesimal formulation of Proposition 1; here we are also able to prove an analogue of the “strong form” of the classical Schwarz lemma (cf. [3]). For every $p \in M$ we denote by $M_p$ the space of real tangent vectors at $p$. $M_p$ is a complex Euclidean space under the hermitian structure of $M$, we denote the length of a vector $X \in M_p$ by $\|X\|$. 

**Proposition 2.** Let $M$ be a hermitian symmetric space of rank $l$ and let $f : M \to M$ be a holomorphic function. Then for all $p \in M$ and $X \in M_p$, we have $\|df(X)\| \leq l^{1/2} \|X\|$, the constant $l^{1/2}$ being the best possible.

If there exists a point $p \in M$ such that $\|df(X)\| > \|X\|$ for all $X \in M_p$, then $f$ is a holomorphic automorphism of $M$.

**Proof.** The first statement follows from Proposition 1. To prove the second statement, let $g \in G$ be such that $gp = f(p)$. Then $h = g^{-1} \circ f$ maps $M$ onto itself and keeps $p$ fixed; since $g^{-1}$ is an isometry, the hypothesis implies $\|dh(X)\| \geq \|X\|$ for all $X \in M_p$, and hence $|\det(dh)_p| \geq 1$. By a well-known theorem of H. Cartan and Carathéodory (e.g. [1, Chapter 1]) it follows that $h$, and therefore also $f$, is a holomorphic automorphism of $M$, finishing the proof.

If $l = 1$, we have the following sharper version of the “strong form.”

**Proposition 3.** Let $M$ be a hermitian symmetric space of rank $1$ and dimension $n$, and let $f : M \to M$ be a holomorphic function. If there exists a point $p \in M$ and $n$ complex-linearly independent vectors $X_1, \ldots, X_n \in M_p$ such that $\|df(X_i)\| = \|X_i\|$ $(i = 1, \ldots, n)$, then $f$ is a holomorphic automorphism of $M$.

**Proof.** Let $g \in G$ be such that $gp = f(p)$, and let $h = g^{-1} \circ f$. By Proposition 2, $(dh)_p$ is a linear contraction of the complex Euclidean space $M_p$. Denoting the adjoint transformation by $(dh)^*_p$ it follows that $A = I - (dh)_p(dh)^*_p$ is a positive semidefinite linear transformation on $M_p$. By our hypothesis $\|AX_i\| = 0$ $(i = 1, \ldots, n)$; this now implies $A = 0$. It follows that $(dh)_p$ is unitary, whence $|\det(dh)_p| = 1$, and by
the above mentioned theorem of Cartan and Carathéodory the proof is finished.

References


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A CHARACTERIZATION OF TAME 2-SPHERES IN $E^3$

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In this note, the tame 2-spheres in $E^3$ are characterized partly in terms of homology and the arcs they contain. In a similar way, the compact 2-manifolds with boundary are characterized. If $K$ is a finite topological 2-complex in $E^3$ and $v$ is a vertex of $K$, then $St v$ is the star of $v$, $\hat{St} v$ is the open star of $v$, and $Lk v = St v - \hat{St} v$ is the link of $v$. The trivial 1-dimensional homology group of $K$ will be denoted by $H_1(K) = 0$.

An $n$-manifold with boundary is a separable metric space such that each point has a neighborhood whose closure is topologically equivalent to a closed $n$-cell.

Theorem 1. Let $K$ be a finite topological 2-complex in $E^3$ such that

(i) $K$ is connected,
(ii) $Lk v$ is connected for each vertex $v$ in $K$,
(iii) $H_1(K) = 0$, and
(iv) $K$ contains only tame arcs.

Then $K$ is either a disk or a 2-sphere.

Proof. Since $K$ contains no wild arcs and $Lk v$ is connected, each 1-simplex in $K$ lies on exactly one or two 2-simplices in $K$ [2]. Since

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1 These results form a part of the author's doctoral dissertation written at the Virginia Polytechnic Institute in 1964 under the direction of Professor P. H. Doyle.