ON THE SERIES OF PRIME RECIPROCALS

JAMES A. CLARKSON

Let $p_n$ be the $n$th prime. We give another proof of the

Theorem. The series $\sum_{n=1}^{\infty} (1/p_n)$ diverges.

Proof. Assume the contrary, and fix $k$ so that

$$\sum_{n=k+1}^{\infty} (1/p_n) < 1/2.$$  (1)

Let $Q = p_1 p_2 \cdots p_k$.

We consider now the sum $S(r) = \sum_{i=1}^r [1/(1+iQ)]$, where $r$ is any
positive integer. Since $1+iQ$ is prime to $Q$, all the prime factors of
all these denominators are from a finite segment of primes which we
 call $P(r)$:

$$P(r) = \{p_{k+1}, p_{k+2}, \ldots, p_{m(r)}\}.$$ 

Now let $S(r, j)$ stand for the sum of those terms in the sum $S(r)$
whose denominators $1+iQ$ have just $j$ (not assumed distinct) prime
 factors. Each such term has the form $1/q_1 q_2 \cdots q_i$, with each
$q_i \in P(r)$. But every such term occurs at least once in the expansion
of $[\sum_{n=k+1}^{m(r)} (1/p_n)]^i$, so by (1) $S(r, j) < 1/2j$. Thus for each $r$,

$$S(r) = \sum_j S(r, j) < \sum_j (1/2j) < 1.$$ 

So $\sum_{n=1}^{\infty} [1/(1+iQ)]$ converges, which in turn implies that the
harmonic series does.

Tufts University

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