ON THE SERIES OF PRIME RECIPROCALS

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Let \( p_n \) be the \( n \)th prime. We give another proof of the theorem. The series \( \sum_{n=1}^{\infty} (1/p_n) \) diverges.

Proof. Assume the contrary, and fix \( k \) so that

\[
\sum_{n=k+1}^{\infty} (1/p_n) < 1/2.
\]

Let \( Q = p_1 p_2 \cdots p_k \).

We consider now the sum \( S(r) = \sum_{i=1}^{r} [1/(1+iQ)] \), where \( r \) is any positive integer. Since \( 1+iQ \) is prime to \( Q \), all the prime factors of all these denominators are from a finite segment of primes which we call \( P(r) \):

\[
P(r) = \{ p_{k+1}, p_{k+2}, \ldots, p_{m(r)} \}.
\]

Now let \( S(r, j) \) stand for the sum of those terms in the sum \( S(r) \) whose denominators \( 1+iQ \) have just \( j \) (not assumed distinct) prime factors. Each such term has the form \( 1/q_1 q_2 \cdots q_j \), with each \( q_i \in P(r) \). But every such term occurs at least once in the expansion of \[ \sum_{n=k+1}^{m(r)} (1/p_n) \] \( j \), so by (1) \( S(r, j) < 1/2^j \). Thus for each \( r \),

\[
S(r) = \sum_j S(r, j) < \sum_j (1/2^j) < 1.
\]

So \( \sum_{r=1}^{\infty} [1/(1+iQ)] \) converges, which in turn implies that the harmonic series does.

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