AN ELEMENTARY NUMBER THEORETICAL REMARK

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For a real number r we define $((r)) = r - [r]$, where $[r]$ is the greatest integer $\leqq r$. Then the number $e = \sum_{n=0}^{\infty} 1/n!$ has, as appears from this representation, the property that the values $((n!\cdot e))$ converge to 0 monotonically, where specially $((n!\cdot e)) < 1/n$ holds $(n>0)$. This elementary fact can be imbedded into a general theorem on real numbers, because the following holds:

THEOREM. *Let r be a nonnegative real number. Then there exists a sequence $\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots$, of nonnegative integers, such that the sequence*

$$a_1 = ((r\cdot 1^{\alpha_1})), \quad a_2 = ((r\cdot 1^{\alpha_1}\cdot 2^{\alpha_2})), \cdots,$$
$$a_n = ((r\cdot 1^{\alpha_1}\cdot 2^{\alpha_2}\cdot 3^{\alpha_3}\cdot \,\cdots\, \cdot n^{\alpha_n})), \cdots$$

converges to 0 monotonically (in the wide sense), where especially the inequality $((r\cdot 1^{\alpha_1}\cdot 2^{\alpha_2}\cdot\,\cdots\,\cdot n^{\alpha_n})) < 1/n$ holds.

PROOF. We put $\alpha_1 = 1$. If we have already defined $\alpha_1, \cdots, \alpha_n$, such that $a_n < 1/n$, it follows from this and from the formula

$$\frac{1}{n} = \sum_{\nu=1}^{\infty} \frac{1}{(n-1)^{\nu}}$$

either $a_n < 1/(n+1)$ or the existence of exactly one k such that

$$\frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+1)^k}$$
$$\leqq a_n < \frac{1}{n+1} + \frac{1}{(n+1)^2} + \cdots + \frac{1}{(n+1)^k} + \frac{1}{(n+1)^{k+1}}\,.$$

In the first case we put $\alpha_{n+1} = 0$, in the second $\alpha_{n+1} = k$. Then it follows in both cases $a_{n+1} = ((a_n\cdot(n+1)^{\alpha_{n+1}})) < 1/(n+1)$, such that the theorem is proved by induction.

If we associate in this way to every nonnegative real number r the corresponding sequence $\alpha_1, \alpha_2, \cdots$, we can easily see, that r is rational if and only if there exists a number ν, such that $\alpha_\mu = 0$ for all $\mu > \nu$. Now to e there corresponds the sequence, where all α_ν are equal 1.

Conversely it can be verified by the method of nested intervals

Received by the editors September 28, 1962.

that to every sequence $\alpha_1, \alpha_2, \cdots$ (with $\alpha_1 = 1$) of nonnegative integers there exists exactly one real number in the interval $[0, 1)$ to which the given sequence $\alpha_1, \alpha_2, \cdots$ corresponds.

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A SHORT PROOF OF AN INEQUALITY FOR THE PERMANENT FUNCTION

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Let A be a substochastic matrix, i.e., a square matrix of nonnegative numbers with each row sum no greater than 1. We have obtained a lower bound for the permanent of $I-A$.

THEOREM. *If A is a substochastic matrix, then*

$$\operatorname{per}(I - A) \geqq 0.$$

It was brought to our attention by Marcus and Minc [2] that Brualdi and Newman have proved this theorem. Indeed, two proofs of this theorem are contained in a paper that will appear in the Oxford Quarterly [1]. The proof that we shall give, shorter than and quite different from the Brualdi-Newman proofs, shows that this theorem is almost a corollary of the Ryser representation of the permanent.

Let B be an n-square matrix and let B_r denote a matrix obtained from B by replacing some r columns of B by zero columns. Let $S(B_r)$ be the product of the row sums of the matrix B_r. Ryser [3] has proved that the permanent of B is given by

$$\operatorname{per}(B) = S(B_0) + \sum (-1)S(B_1) + \sum (-1)^2 S(B_2) + \cdots + \sum (-1)^{n-1} S(B_{n-1}),$$

where $\sum(-1)^r S(B_r)$ denotes the sum over all $\binom{n}{r}$ replacements of r of the columns of B by zero columns.

Let $B = I - A$ where A is a substochastic matrix. The ith row sum of B_r is nonpositive or nonnegative according to whether the ith column of B_r is a zero or a nonzero column. Hence there are at least r row sums of B_r that are nonpositive and at least $n-r$ that are nonnegative. Therefore

Received by the editors September 2, 1965.