ON THE DIOPHANTINE EQUATION \( x^3 + y^3 + z^3 = x + y + z \)

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1. The remarks in this note on the Diophantine equation

\[ x^3 + y^3 + z^3 = x + y + z \]

are prompted by Edgar's recent note [1]. In order "to avoid certain
trivial solutions" he assumes that \( x \geq y \geq 0, \ z < 0 \) and \( x \neq -y \). Using a
method of S. D. Chowla and others (a reference not accessible to me)
he obtains infinitely many solutions of (1) subject to the further con-
ditions

\[ x + y + z = m, \quad x + y = km, \quad x + z \neq 0 \]
in each of the following cases (i) \( k = 3 \) (Chowla), (ii) \( k = 12 \) (Edgar),
(iii) \( k = 16/3 \) (Edgar).

In this note I show that each of the trivial solutions \((h, 1, -h)\)
where \(|h| \geq 2\), gives rise to infinitely many nontrivial solutions and
that nontrivial solutions likewise generate others.

As an example the equation

\[ N^2 - 85M^2 = -4 \]

has infinitely many integral solutions \((N, M)\), both odd or both
even. The integers

\[ x = \frac{1}{2}(M + N), \quad y = \frac{1}{2}(M - N), \quad z = -4M \]

will always satisfy (1). And for those solutions

\[ x + y + z = -3M, \quad x + y = M. \]

These solutions were obtained in fact by the method below from the
nontrivial solution \((5, -4, -4)\).

The equation

\[ 3N^2 - 31M^2 = -4 \]

also has infinitely many solutions with \( M, N \) of like parity: the equa-
tions

\[ x + y = -M, \quad x - y = N, \quad z = 2M \]

will yield solutions of (1). Equation (5) was derived from the trivial
solution \((-2, 1, 2)\) of (1).

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(6) \[ 5N^2 - 62M^2 = 2 \]

has infinitely many solutions in integers \( N \)(even) and \( M \): the smallest solution appears to be \((412, 117)\). Determine \( x, y, z \) by the equations

\[
\begin{align*}
x + y &= 10M, \\
x - y &= N, \\
z &= -7M;
\end{align*}
\]

then \( x, y, z \) are integers which satisfy (1). One solution is therefore \( x = 791, y = 379, z = -819 \).

2. Suppose that \((x, y, z)\) is a solution of (1). Any permutation yields another solution (not necessarily distinct). Also \((-x, -y, -z)\) is a solution. In general 12 solutions arise from a given solution.

Suppose that \(x + y\) and \(z\) are not both zero. Define integers \(m, n, a, c\) uniquely by the following equations

\[(7) \quad x + y = am, \quad z = -cm, \quad x - y = n, \quad (a, c) = 1, \quad m \geq 1.\]

Then from the identity

\[
4(x^3 + y^3 + z^3 - x - y - z) = m\left\{3an^2 + (a^3 - 4c^3)m^2 - 4(a - c)\right\}
\]

we see that the integers \((m, n)\) \((m \geq 1)\) satisfy the Diophantine equation

\[
(a^3 - 4c^3)M^2 + 3aN^2 = 4(a - c). \tag{8}
\]

Conversely, suppose that integers \(a\) and \(c\) exist such that (8) is solvable in integers \(M \neq 0, N\) with \(aM, N\) of like parity; then the equations

\[
X + Y = aM, \quad Z = -cM, \quad X - Y = N
\]

give integers \(X, Y, Z\) which satisfy (1).

If in addition the integer \(D\) defined by

\[
D = 3a(4c^3 - a^3) \tag{9}
\]

is positive and not a square, then the equation (8), having one solution \((M, N)\) with \(aM, N\) of same parity, will have infinitely many such solutions, by a classical theorem on indefinite binary quadratic forms.

As an example, take the trivial solution

\[
(x, y, z) = (h, 1, -h)
\]

where \(h\) is an integer, \(|h| \geq 2\). Equation (8) becomes

\[
3(h + 1)N^2 - (3h^3 - 3h^2 - 3h - 1)M^2 = 4,
\]

\[
D = 3(h + 1)(3h^3 - 3h^2 - 3h - 1). \tag{10}
\]
It is easily seen that $D>0$ and that $D$ is not a square whenever $|h| \geq 2$. Now (10) has the solution $M=1$, $N=h-1$: it has therefore infinitely many solutions such that $(h+1)M$, $N$ have the same parity. I omit the proof.

Here are a few examples of (10):

\[
\begin{align*}
9N^2 - 5M^2 &= 4, & 31M^2 - 3N^2 &= 4, \\
3N^2 - 11M^2 &= 1, & 100M^2 - 6N^2 &= 4, \\
15N^2 - 131M^2 &= 4, & 229M^2 - 9N^2 &= 4, \\
18N^2 - 284M^2 &= 4, & 109M^2 - 3N^2 &= 1.
\end{align*}
\]

3. Each solution $(x, y, z)$ of (1) gives rise in general to three pairs of integers $(a, c)$ and hence to three binary forms. Suppose $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are two solutions of (1) derived from two pairs $(N_1, M_1)$, $(N_2, M_2)$, belonging to a particular binary form corresponding to a pair $(a, c)$: suppose also that $(x_1, y_1, z_1) \neq (x_2, y_2, z_2)$ or to $(-x_2, -y_2, -z_2)$. Then the two remaining binary forms deducible from the triad $(x_1, y_1, z_1)$ are distinct from those deducible from the triad $(x_2, y_2, z_2)$. In this way further sets of solutions can be generated.

As an example the trivial solution $(2, 1, -2)$ leads to the form $9N^2 - 5M^2 = 4$. The solution $(6, 8)$ of the latter equation gives the triad $(15, 9, -16)$ which satisfies (1). This triad yields the triads $(15, -16, 9), (9, -16, 15)$ whence we get two sets for $a, c, m, n$ and two forms:

\[
\begin{align*}
-1, -9, 1, 31: & \quad 2915M^2 - 3N^2 = 32; \\
-7, -15, 1, 25: & \quad 13157M^2 - 21N^2 = 32.
\end{align*}
\]

From these two binary forms infinitely many others can be generated, each of which will lead to solutions of (1).

4. Edgar [1] gives a solution of (1) corresponding in his notation to $k = 16/3$; in my notation $a = 16, c = 13$. The corresponding equation (8) is

\[
4N^2 - 391M^2 = 1
\]

which has (as Edgar says) infinitely many solutions with even $N$, the smallest solution yielding

\[
x = 8u + v, \quad y = 8u - v, \quad z = 13u
\]

where $u = 371133, v = 1834670$. In the way described further forms can be generated from the permutations

\[
(8u + v, -13u, 8u - v), \quad (8u - v, -13u, 8u + v).
\]
The pair \((a, c) = (10, 7)\) is also worthy of note: it leads to the form (6) which gives rise to infinitely many solutions of (1).

Two more examples can be given of small \(a, c\):

\[
\begin{align*}
a &= 14, & c &= 11; \\
a &= 64, & c &= 61.
\end{align*}
\]

The first leads to the equation

\[
7N^2 - 430M^2 = 2,
\]

solvable infinitely often with \(N\) even, e.g. \(N = 2124, M = 271\) whence

\[
x = 2959, \quad y = 835, \quad z = -2981.
\]

The second \((64, 61)\) leads to the equation

\[
(4N)^2 - 53815M^2 = 1
\]

which is in fact solvably infinitely often with \(N\) even so that solutions of (1) are given by

\[
x + y = 64M, \quad x - y = N, \quad z = -61M.
\]

5. A difficult problem remains for consideration. Two solutions of (1) may be regarded as dependent if they can be connected by a finite number of binary forms as described above. Can simple criteria be determined for dependence? Can the independent solutions be completely specified?

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**Reference**


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