APPROXIMATE STOLZ ANGLE LIMITS

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1. **Introduction.** In a recent paper [4] the author discussed functions of a real variable that are obtained as Soltz angle limits of functions of two real variables. The central idea of this paper is the weakening of the hypothesis of existence of limits with respect to Stolz angles to existence of approximate limits with respect to Stolz angles. A consequence of these investigations is another proof of the fact that an approximate derivative is in the first Baire class. This was proved by Tolstoff [5] in 1938 and more recently Goffman and Neugebauer [2] gave a less involved proof than that of Tolstoff.

2. **Definitions and notation.** By a Stolz angle we mean an angular sector in the upper half-plane with its vertex on the x-axis. If S is such a Stolz angle and r a positive number, then S^r denotes the set of points in S which lie on or below the line $y=1/r$. $|E|$ denotes either the 1-dimensional or 2-dimensional Lebesgue measure of the set E. It will be clear from the context which is intended.

Let S_x be a Soltz angle with vertex $(x, 0)$ and let E be a measurable set in the upper half-plane. The lower metric density of E at $(x, 0)$ relative to S_x is

$$\liminf_{r\to\infty} \frac{|S_x^r \cap E|}{|S_x^r|}.$$

If this number is 1, the point $(x, 0)$ is said to be a point of density of E relative to S_x.

A function f of one real variable is called a boundary function of a function ϕ defined on the upper half-plane if for each x, $f(x)$ is the limit of $\phi(u, v)$ as (u, v) approaches $(x, 0)$ relative to some set in the upper half-plane with $(x, 0)$ as a limit point.

The symbol $\omega(x, f)$ denotes the oscillation of the function f at the point x.

The symbol $l(x, \theta)$ denotes the half-line from the point $(x, 0)$ whose angle of inclination is θ. R denotes the reals and W denotes the open upper half-plane.

3. **Main theorem.** In this section S_x is used to denote the Stolz angle with vertex $(x, 0)$ which consists of the angular sector between $l(x, \pi/4)$ and $l(x, 3\pi/4)$.

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THEOREM 1. *Suppose* ϕ: $W \to R$ *and that for each* $x \in R$ *there is a set* $E_x \subset W$ *such that*

(i) $(x, 0)$ *is a point of density of* E_x *relative to* S_x *and*

(ii) $\lim \phi(u, v)$ *exists as* $(u, v) \to (x, 0)$ *relative to the set* E_x, *for each* $x \in R$. *Then the boundary function of* ϕ *determined by the family of sets* $\{E_x\}$ *is in the first Baire class.*

PROOF. Let f denote the boundary function. For the proof we show that for any nonempty perfect set P, $f|P$ has a point of continuity so that by Baire's theorem [3], f is in the first Baire class.

Suppose P is a nonempty perfect set such that $f|P$ has no point of continuity. Let $D_n = \{x \in P: \omega(x, f|P) \geqq 1/n\}$. It follows that $P = \bigcup_{n=1}^{\infty} D_n$. Since each D_n is closed and P is of the second category in itself, there is an n_0 and an open interval I such that $I \cap P \neq \emptyset$ and D_{n_0} contains $I \cap P$. Let $Q = \mathrm{cl}(I \cap P)$. The set Q is perfect and $\omega(x, f|Q) \geqq 1/n_0$ on $I \cap P$.

Let $E_x^n = S_x^n \cap E_x$, where S_x^n is the set $\{(u, v) \in S_x: v \leqq 1/n\}$. For each positive integer k, let A_k be the set defined by

$$A_k = \{x \in Q: |E_x^n| > 7/8 \cdot |S_x^n| \text{ for } n \geqq k \text{ and } p \in E_x^k \Rightarrow |\phi(p) - f(x)| < 1/M\},$$

where $M = 16n_0$. Note that $A_k \subset A_{k+1}$. It also follows that $Q = \bigcup_{k=1}^{\infty} A_k$ since each point $(x, 0)$ is a point of density of E_x relative to S_x and $f(x)$ is the boundary limit of ϕ relative to E_x. Since Q is of the second category in itself, there is an integer q and an open interval J such that $J \cap Q \neq \emptyset$ and A_q is dense in $J \cap Q$.

Let x_0 be a fixed point in $A_q \cap J$. Note that if $|x_0 - x| < 1/q$, then $|S_{x_0}^q \cap S_x^q| > 1/4q^2 = |S_{x_0}^q|/4$. Thus if $x \in A_q \cap (x_0 - 1/q, x_0 + 1/q)$, then $E_{x_0}^q \cap E_x^q \neq \emptyset$.

Let $\{x_j\}$ be any sequence of points in Q with x_0 as limit. There is an integer K such that $j > K$ implies that $|x_0 - x_j| < 1/q$ and $x_j \in J$. Fix such a j and let $r = \max\{q, k(x_j)\}$, where $k(x_j)$ is the smallest positive integer k such that $x_j \in A_k$. Since A_q is dense in $J \cap Q$, there is a point y in $A_q \cap J$ such that $|y - x_j| < 1/r$. It follows that $E_y^r \cap E_{x_j}^r \neq \emptyset$ and $E_y^q \cap E_{x_0}^q \neq \emptyset$. Let $\xi \in E_y^r \cap E_{x_j}^r$ and $\eta \in E_y^q \cap E_{x_0}^q$. Then

$$\begin{aligned} |f(x_j) - f(x_0)| &\leqq |f(x_j) - \phi(\xi)| + |\phi(\xi) - f(y)| \\ &\quad + |f(y) - \phi(\eta)| + |\phi(\eta) - f(x_0)| \\ &< 4/M \\ &= 1/4n_0. \end{aligned}$$

But this implies that $\omega(x_0, f|Q) \leqq 1/2n_0$ which is a contradiction.

Therefore $f|P$ has a point of continuity for any nonempty perfect set P.

4. **Noncongruent Stolz angles.** A function f: $R\to R$ is an honorary function of the second class if there is a function g in the first Baire class such that $f(x)=g(x)$ except on a countable set. (This was introduced by Bagemihl and Piranian [**1**].)

In Theorem 1 all of the Stolz angles were congruent and each was symmetric about the vertical line through its vertex. This seems to be a severe restriction and, as we shall see later, is more than is necessary. However, if we drop the requirement that the Stolz angles be congruent and require only that each Stolz angle S_x be symmetric about the half-line $l(x, \pi/2)$, then the boundary function may be in the second Baire class even though the function ϕ be continuous. This is illustrated by the following theorem.

THEOREM 2. *If f: $R\to R$ is any honorary function of the second Baire class, then there is a continuous function ϕ: $W\to R$ such that f is a boundary function of ϕ obtained by approximate limits relative to Stolz angles S_x symmetric about the half-lines $l(x, \pi/2)$.*

PROOF. Let g: $R\to R$ be a function in the first Baire class which is equal to f except for a countable set. By a theorem in [**4**], there is a continuous function φ: $W\to R$ such that $\lim \varphi(u, v)=g(x)$ as $(u, v)\to(x, 0)$ relative to any Stolz angle with vertex $(x, 0)$, i.e., for any nontangential approach to $(x, 0)$, the limit of φ exists and is equal to $g(x)$. Let $\{r_n\}$ be a sequential ordering of the exceptional points. For each x, let S_x denote the Stolz angle which has as its sides the half-lines $l(x, \pi/4)$ and $l(x, 3\pi/4)$. For $x=r_n$, let T_x be a Stolz angle which is symmetric about $l(x, \pi/2)$ and satisfies $|T_x^a|=|S_x^a|\cdot(1/2^n)$ for every positive real number a. For each $n=1, 2, \cdots$, it is possible to truncate T_{r_n} at some height, say $1/k$, so that $T_{r_n}^k\cap\bigcup_{i=1}^{n-1} T_{r_i}$ is empty. Let us suppose that this has been done for each $x=r_n$ and let us denote the truncated Stolz angles by T_x likewise. Next let $\tilde{T}_x$ be a truncated Stolz angle contained in the interior of T_x except that T_x and $\tilde{T}_x$ have their vertex in common.

We define ϕ: $W\to R$ as follows: let $\phi(u, v)=\varphi(u, v)$ for any point (u, v) not in $\bigcup_{n=1}^{\infty} \operatorname{Int}(T_{r_n})$, let $\phi(u, v)=f(x)$ for every point in $\tilde{T}_x$ ($x=r_n$ for some n), and extend continuously over the remainder of T_{r_n} for each n.

For $x=r_n$, the set E_x on which ϕ has the proper boundary limit is the set $\tilde{T}_x$. Then clearly $(x, 0)$ is a point of density of E_x relative to the Stolz angle $\tilde{T}_x$. For a point x which is not one of the r_n's, the set

E_x is S_x minus the union of the T_{r_n}'s. It is clear from the definition of ϕ that ϕ has the proper boundary limit relative to the set E_x. It only remains to verify that $(x, 0)$ is a point of density of E_x relative to S_x. Let $\epsilon > 0$ be given. There is an integer N such that $n > N$ implies $\sum_{k=1}^{\infty} 1/2^k < \epsilon$. Let M be chosen so large that $m \geqq M$ implies that $S_x^m \cap \bigcup_{n=1}^{N} T_{r_n} = \emptyset$. For $m \geqq M$,

$$\frac{|E_x^m|}{|S_x^m|} = \frac{\left| S_x^m - \bigcup_{n=N+1} T_{r_n}^m \right|}{|S_x^m|} \geqq 1 - \sum_{n=N+1}^{\infty} \frac{|T_{r_n}^m|}{|S_x^m|}$$
$$= 1 - \sum_{n=N+1}^{\infty} \frac{1}{2^n} > 1 - \epsilon.$$

This verifies the density of E_x at $(x, 0)$ relative to S_x and completes the proof.

In [4] it was shown that the class of functions which are Stolz angle boundary functions is a proper subclass of the class of honorary functions of the second Baire class. Here we note that if a function is an approximate Stolz angle boundary function, with the Stolz angles being suitably restricted, then the function is in the first Baire class, whereas if the restrictions on the Stolz angles are slightly weakened, then any honorary function can be realized as the approximate Stolz angle boundary function of a continuous function.

5. **Corollaries of Theorem** 1. For each $x \in R$, let S_x again denote the Stolz angle whose sides are $l(x, \pi/4)$ and $l(x, 3\pi/4)$. In the following corollary it is shown that the hypothesis on the E_x's can be weakened without changing the conclusion of Theorem 1.

COROLLARY 1. *Theorem* 1 *remains valid if in place of* (i) *it is only required that there is a constant* $K > 1/2$ *such that the lower metric density of* E_x *at* $(x, 0)$ *relative to* S_x *is greater than or equal to* K.

In order to prove Corollary 1 it is only necessary to make a few modifications in the proof of Theorem 1. For example, define A_k by

$$A_k = \{x \in Q: |E_x^n| > \zeta \cdot |S_x^n| \text{ for } n \geqq k \text{ and } p \in E_x^k \Rightarrow$$
$$|\phi(p) - f(x)| < 1/M\},$$

where $1/2 < \zeta < K$ and $M = 16n_0$. At a later step, assuming q has been selected as before, it is necessary to find a positive number δ so that $|x_0 - x| < \delta$ implies $|S_{x_0}^q \cap S_x^q| > 2 \cdot (1 - \eta) \cdot |S_{x_0}^q|$. From the previous inequality it follows that $E_{x_0}^q \cap E_x^q \neq \emptyset$ for $x \in A_q \cap (x_0 - \delta, x_0 + \delta)$. The remainder of the proof is essentially the same.

Another direction to proceed in considering weakenings of the hypotheses of Theorem 1 is to allow more general Stolz angles S_x, that is, to allow the "size" of S_x to vary with x. Of course, as can be seen in §4, it will be necessary to impose some restrictions on the varying in order to obtain the same conclusion as in Theorem 1.

We will continue to consider only Stolz angles S_x that are symmetric about the line $l(x, \pi/2)$. Let us suppose that a family of Stolz angles $\{S_x: x \in R\}$ is given. Let $\theta(x)$ denote the angle between the sides of S_x. Then corresponding to the family $\{S_x: x \in R\}$ there is a function $\theta: R \to (0, \pi)$.

COROLLARY 2. *Suppose ϕ: $W \to R$ and that for each $x \in R$ there is a set $E_x \subset W$ such that*

(i) $(x, 0)$ *is a point of density of E_x relative to S_x and*

(ii) $\lim \phi(u, v)$ *exists as* $(u, v) \to (x, 0)$ *relative to the set E_x, for each $x \in R$. If the function $\theta: R \to (0, \pi)$ associated with the family of Stolz angles is upper semicontinuous, then the boundary function f of ϕ determined by the family of sets E_x: $X \in R$ is in the first Baire class.*

PROOF. Suppose that there is a nonempty perfect set P for which $f|P$ has no point of continuity. For each $n = 1, 2, \cdots$, let $B_n = \{x \in P: \theta(x) \geqq 1/n\}$. Since θ is upper semicontinuous, each B_n is closed. Also $P = \bigcup_{n=1}^{\infty} B_n$. Since P is second category in itself, there is an integer n_1 and an open interval J such that B_{n_1} is dense in $J \cap P$ and $J \cap P \neq \emptyset$. From this it follows that $B_{n_1} \supset J \cap P$. Then $\theta(x) \geqq 1/n_1$ for $x \in J \cap P$. Then for each $x \in J \cap P$, let T_x be a symmetric Stolz angle with vertex at $(x, 0)$ and with the angle between its sides being $1/n_1$. The point $(x, 0)$ is also a point of density of E_x relative to T_x.

The remainder of the proof consists of showing that $f|P$ has a point of continuity in $J \cap P$. This is accomplished by the same means used in the proof of Theorem 1, using the T_x's instead of the original S_x's. We omit the details of this.

6. **Approximate derivatives.** Let I_0 be an open interval and let $f: I_0 \to R$. The number A is called the *approximate derivative* of the function f at x_0 if there exists a set E, having x_0 as a point of metric density one, such that for $x \in E$ and $x \to x_0$ we have

$$\lim(f(x) - f(x_0))/(x - x_0) = A.$$

In this case we use the symbol $f'_{ap}(x_0)$ for the number A. In this section we are interested in the function f'_{ap} whenever it exists everywhere on some interval.

THEOREM 3. *Let $f: I_0 \to R$. If $f'_{ap}(x)$ exists for each $x \in I_0$, then the function f'_{ap} is in the first Baire class.*

PROOF. For the proof we construct a function $\phi: I_0 \times (0, \infty) \to R$ which has f'_{ap} as its approximate Stolz angle boundary function (in the sense of Theorem 1).

Let U denote $I_0 \times (0, \infty)$. If $(x, r) \in U$ and if $x - r/2$ and $x + r/2$ are both in I_0, let $\phi(x, r) = (f(x + r/2) - f(x - r/2))/r$. If $(x, r) \in U$ and if either $x - r/2$ or $x + r/2$ is not in I_0, let $\phi(x, r) = 0$.

Let $x_0 \in I_0$ be fixed. We will show that there is a set $E_{x_0} \subset U$ such that $(x_0, 0)$ is a point of density of E_{x_0} relative to S_{x_0} and such that $\lim \phi(x, r) = f(x_0)$ for $(x, r) \in E_{x_0}$ and $(x, r) \to (x_0, 0)$. By hypothesis there is a set $B_{x_0} \subset I_0$ such that the linear density of B_{x_0} at x_0 is 1 and $\lim (f(x) - f(x_0))/(x - x_0) = f'_{ap}(x_0)$ for $x \in B_{x_0}$ and $x \to x_0$. Let E_{x_0} be the set defined by $E_{x_0} = \{(x, r) \in U: x_0 - r/2 \leqq x \leqq x_0 + r/2,\ x + r/2 \in B_{x_0}$, and $x - r/2 \in B_{x_0}\}$, and let $\{(x_n, r_n)\}$ be a sequence of points in E_{x_0} with $(x_0, 0)$ as limit. We have

$$\left| \phi(x_n, r_n) - f'_{ap}(x_0) \right| = \left| \frac{f(x_n + r_n/2) - f(x_n - r_n/2)}{r_n} - f'_{ap}(x_0) \right|$$

$$= \left| \left(\frac{f(x_n + r_n/2) - f(x_0)}{x_n + r_n/2 - x_0} - f'_{ap}(x_0) \right) \cdot \left(\frac{x_n + r_n/2 - x_0}{r_n} \right) \right.$$

$$\left. + \left(\frac{f(x_0) - f(x_n - r_n/2)}{x_0 - x_n + r_n/2} - f'_{ap}(x_0) \right) \cdot \left(\frac{x_0 - x_n + r_n/2}{r_n} \right) \right|$$

$$\leqq \left| \frac{f(x_n + r_n/2) - f(x_0)}{x_n + r_n/2 - x_0} - f'_{ap}(x_0) \right| + \left| \frac{f(x_0) - f(x_n - r_n/2)}{x_0 - x_n + r_n/2} - f'_{ap}(x_0) \right|.$$

The two expressions on the extreme right of the inequality approach zero as $n \to \infty$ since $x_n - r_n/2$ and $x_n + r_n/2$ are in B_{x_0}. Therefore $\phi(x_n, r_n) \to f'_{ap}(x_0)$.

It remains to verify the density property of E_{x_0}. For this end, let ϵ be an arbitrary positive number. Since B_{x_0} has x_0 as a point of density, there is a number $R(\epsilon)$ such that $r < R(\epsilon)$ implies that $|B_{x_0} \cap (x_0 - r, x_0 + r)| > (1 - \epsilon) \cdot 2r$. Let r less than $R(\epsilon)$ be fixed, and let η denote the linear Lebesgue measure of the intersection of E_{x_0} with the line $y = r/2$. Let $D = \{x \in (x_0 - r/2, x_0 + r/2): x + r/2 \in B_{x_0}$ and $x - r/2 \in B_{x_0}\}$, then $\eta = |D|$. Let $F = (x_0 - r/2, x_0 + r/2) \sim D$. If $x \in F$, then $x + r/2$ or $x - r/2$ is not in B_{x_0}. But the points $x + r/2$ and $x - r/2$ lie in the interval $(x_0 - r, x_0 + r)$ and the measure of the set of points in (x_0-r, x_0+r) which are not in B_{x_0} is less than $\epsilon\cdot 2r$. Thus the measure of F is no more than $\epsilon\cdot 4r$, and so $\eta>(1-2\epsilon)\cdot 2r$. It now follows that for $r<R(\epsilon)$, the relative measure of B_{x_0} in S_{x_0}, truncated at $y=r$, is greater than or equal to $(1-2\epsilon)$ times the measure of the truncated Stolz angle. This completes the proof.

REMARK. Theorem 1 also implies the well-known fact that approximately continuous functions are in the first Baire class. For if f: $R\rightarrow R$ is approximately continuous, let $\phi(x, y)=f(x)$. Clearly f is an approximate Stolz angle boundary function for ϕ in the sense of Theorem 1.

In the theorems above, sets with zero density at certain points were the exceptional sets. Analogous theorems are true if one considers first category sets as the exceptional sets.

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