(v) $$\begin{pmatrix} \|Z\| & 0 \\ 0 & I \end{pmatrix}.$$

It is possible that these together with the constant unitary matrices generate the whole class of such functions, but we have not been able to prove it.

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# ON THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

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Let $\mathcal{C}$ denote the class of functions $f$ regular and univalent in $E=\{z \mid |z|<1\}$, which satisfy $f(0)=0$ and $f'(0)=1$ and which are close-to-convex in $E$. Let $\mathcal{K}$ and $\mathcal{S}^*$ denote the subfamilies of $\mathcal{C}$, made up of functions which are convex and starlike in $E$, respectively. Recently, Libera [2] has shown that if $f$ is a member of $\mathcal{K}$, $\mathcal{S}^*$ or $\mathcal{C}$, then the function $F(z)=(2/z)\int_0^z f(t)dt$ is also a member of $\mathcal{K}$, $\mathcal{S}^*$ or $\mathcal{C}$. It is the purpose of this paper to investigate the converse question. That is, if $F$ is in $\mathcal{S}^*$, what is the radius of starlikeness of the function $f(z)=[1/2][zF(z)]'$? Similar questions are answered under the assumption that $F$ is in $\mathcal{K}$ or in $\mathcal{C}$. Robinson [**5**] has shown that if $F$ is only assumed to be univalent in $E$, then $f$ is starlike for $|z|<.38$. He pointed out that it is probable that $f$ is univalent for $|z|<(1/2)$. We obtain this result under the added assumption that $F$ is a member of $\mathcal{K}$, $\mathcal{S}^*$ or $\mathcal{C}$.

The method of proof used in Theorem 1 has recently been employed by MacGregor [**4**].

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THEOREM 1. *If $F$ is in $\mathcal{S}^*$, then $f(z)=[1/2][zF(z)]'$ is starlike for $|z|<1/2$. This result is sharp.*

PROOF. Since $F$ is in $\mathcal{S}^*$, $\operatorname{Re}[zF'(z)/F(z)]>0$ for $|z|<1$. Thus there exists $\phi$, regular in $E$, such that $|\phi(z)|\leqq 1$ for $z$ in $E$ and such that

$$\frac{zf(z)-\int_0^z f(t)dt}{\int_0^z f(t)dt}=\frac{zF'(z)}{F(z)}=\frac{1-z\phi(z)}{1+z\phi(z)}\cdot$$

Thus

$$f(z)=\frac{2}{z(1+z\phi(z))}\int_0^z f(t)dt.$$

Therefore

$$\frac{zf'(z)}{f(z)}=\frac{-z\phi(z)-z^2\phi'(z)}{1+z\phi(z)}+\frac{zf(z)-\int_0^z f(t)dt}{\int_0^z f(t)dt} \tag{1}$$
$$=\frac{1-2z\phi(z)-z^2\phi'(z)}{1+z\phi(z)}\cdot$$

In order to determine where $f$ is starlike, we must determine those values of $z$ for which the real part of the right hand side of (1) is positive. This condition is equivalent to

$$\operatorname{Re}[1-2z\phi(z)-z^2\phi'(z)][1+z\phi(z)]^- > 0. \tag{2}$$

Condition (2) is equivalent to

$$\operatorname{Re}[z^2\phi'(z)][1+z\phi(z)]^- < 1-2|z|^2|\phi(z)|^2-\operatorname{Re}[z\phi(z)]. \tag{3}$$

Using the well known result

$$|\phi'(z)| \leqq \frac{1}{1-|z|^2}(1-|\phi(z)|^2) \qquad (|z|<1)$$

and using the fact that Re $[z\phi(z)]\leqq|z|\ |\phi(z)|$, we see that condition (3) will be satisfied if

$$\frac{|z|^2}{1-|z|^2}(1-|\phi(z)|^2)(1+|z||\phi(z)|) \tag{4}$$
$$<(1-2|z||\phi(z)|)(1+|z||\phi(z)|).$$

Condition (4) is equivalent to

$$2|z|^2 + 2|z||\phi(z)|(1 - |z|^2) - |z|^2|\phi(z)|^2 < 1. \tag{5}$$

Thus, we need only show that condition (5) holds for all functions $\phi$, regular in $E$ and satisfying $|\phi(z)| \leqq 1$ for $z$ in $E$, provided $|z| < 1/2$.

If in (5) we let $a = |z|$ and $x = |\phi(z)|$, then it is sufficient to show that for any fixed $a$, $0 \leqq a < 1/2$, the function $p(x) = 2a^2 + 2a(1-a^2)x - a^2x^2$ is bounded above by one for $0 \leqq x \leqq 1$. It is easily seen that $p'(x) > 0$, $0 \leqq x \leqq 1$, provided that $a < (\sqrt{5}-1)/2$ and therefore if $a < 1/2$. Thus, if $0 \leqq a < 1/2$, the maximum value of $p(x)$, $0 \leqq x \leqq 1$, is given by $q(a) = 2a + a^2 - 2a^3$. Since $q'(a) > 0$ for $0 \leqq a < 1/2$, $q(a) < q(1/2) = 1$ for $0 \leqq a < 1/2$. Condition (2) is thus seen to be satisfied, if $|z| < 1/2$. Hence $f$ is starlike for $|z| < 1/2$.

To see that the result is sharp, let $F(z) = z/(1-z)^2$ which is in $\mathcal{S}^*$. Then, $f(z) = z/(1-z)^3$ and $zf'(z)/f(z) = (1+2z)/(1-z) = 0$ for $z = -1/2$. Thus, $f$ is not starlike in any circle $|z| < r$, if $r > 1/2$.

THEOREM 2. *If $F$ is in $\mathcal{K}$, then $f(z) = [1/2][zF(z)]'$ is univalent in $E$ and is convex for $|z| < 1/2$. This result is sharp.*

PROOF. We have $2f'(z) = 2F'(z) + zF''(z)$. Thus

$$2 \operatorname{Re}\left[\frac{f'(z)}{F'(z)}\right] = 2 + \operatorname{Re}\left[\frac{zF''(z)}{F'(z)}\right]. \tag{6}$$

Since $F$ is in $\mathcal{K}$, the right hand side of (6) is positive in $E$. Thus, $f$ is close-to-convex relative to $F$ and therefore is univalent in $E$.

To show that $f$ is convex for $|z| < 1/2$, we notice that $zf'(z) = [1/2] \cdot [z(zF'(z))]'$. Since $F$ is in $\mathcal{K}$, $zF'$ is in $\mathcal{S}^*$. Therefore, by Theorem 1, $zf'$ is starlike for $|z| < 1/2$ and thus $f$ is convex for $|z| < 1/2$.

To see that the result is sharp, let $F(z) = z/(1-z)$ which is in $\mathcal{K}$. Then $f(z) = (2z - z^2)/2(1-z)^2$ and $1 + [zF''(z)/F'(z)] = (1+2z)/(1-z) = 0$ for $z = -1/2$. Therefore $f$ is not convex in any circle $|z| < r$, if $r > 1/2$.

THEOREM 3. *If $F$ is in $\mathcal{C}$, then $f(z) = 1/2[zF(z)]'$ is close-to-convex for $|z| < 1/2$. This result is sharp.*

PROOF. Since $F$ is in $\mathcal{C}$, there exists $G$ in $\mathcal{S}^*$ such that

$$\operatorname{Re}\left[\frac{zF'(z)}{G(z)}\right] > 0 \qquad (|z| < 1). \tag{7}$$

Let $g(z) = [1/2][zG(z)]'$, then, by Theorem 1, $g$ is starlike for $|z| < 1/2$. To prove the theorem, it is sufficient to show that $\operatorname{Re}[zf'(z)/g(z)] > 0$ for $|z| < 1/2$. We have

$$\frac{zF'(z)}{G(z)} = \frac{zf(z) - \int_0^z f(t)dt}{\int_0^z g(t)dt}.$$

Thus, by (7), we may set

$$\frac{zf(z) - \int_0^z f(t)dt}{\int_0^z g(t)dt} = P(z) \tag{8}$$

where $P$ is regular in $E$ and satisfies $P(0)=1$ and Re $(P(z))>0$ for $z$ in $E$. We thus have

$$zf'(z) = P(z)g(z) + P'(z)\int_0^z g(t)dt. \tag{9}$$

Therefore

$$\frac{zf'(z)}{g(z)} = P(z) + P'(z)\left[\frac{\int_0^z g(t)dt}{g(z)}\right]. \tag{10}$$

Using the known result [1], [3], [6]

$$|P'(z)| \leqq \frac{2\,\mathrm{Re}[P(z)]}{1-|z|^2} \qquad (|z|<1),$$

we have from (10)

$$\mathrm{Re}\left[\frac{zf'(z)}{g(z)}\right] \geqq \mathrm{Re}[P(z)]\left[1 - \frac{2}{1-|z|^2}\left|\frac{\int_0^z g(t)dt}{g(z)}\right|\right]. \tag{11}$$

Moreover

$$\frac{zg(z)}{\int_0^z g(t)dt} = \frac{[1/2](z[zG(z)]')}{[1/2](zG(z))} = 1 + \frac{zG'(z)}{G(z)}.$$

Since $G$ is in $\mathcal{S}^*$, $\mathrm{Re}[zG'(z)/G(z)]>0$ for $z$ in $E$. Thus $\mathrm{Re}[zg(z)/\int_0^z g(t)dt]>1$ for $z$ in $E$. Hence, there exists $\phi$, regular in $E$ and satisfying $|\phi(z)|\leqq 1$ for $z$ in $E$, such that $zg(z)/\int_0^z g(t)dt=$

$2/(1+z\phi(z))$. Therefore

$$\left|\frac{\int_0^z g(t)dt}{g(z)}\right| = \left|\frac{z+z^2\phi(z)}{2}\right| \leqq \frac{1}{2}(|z|+|z|^2). \tag{12}$$

Combining (11) and (12) we have

$$\begin{aligned} \operatorname{Re}\left[\frac{zf'(z)}{g(z)}\right] &> \operatorname{Re}[P(z)]\left[1-\frac{|z|+|z|^2}{1-|z|^2}\right] \\ &= \operatorname{Re}[P(z)]\left[\frac{1-2|z|}{1-|z|}\right]. \end{aligned} \tag{13}$$

The right hand side of (13) is positive provided $|z|<1/2$.

To see that the result is sharp, let $F(z)=z/(1-z)^2$ which is in $\mathcal{S}^*$ and therefore in $\mathcal{C}$. Then $f(z)=z/(1-z)^3$ and $f'(z)=(1+2z)/(1-z)^4 =0$ for $z=-1/2$. Thus, $f(z)$ is not univalent and therefore not close-to-convex in $|z|<r$, if $r>1/2$.

An interesting subclass of $\mathcal{C}$ is that class made up of functions $F$ which satisfy $\operatorname{Re}[F'(z)]>0$ for $z$ in $E$ [**3**]. Theorem 3 can be improved for this subclass.

THEOREM 4. *Let $F$ be such that* $\operatorname{Re}[F'(z)]>0$ *for $z$ in $E$ and let* $f(z)=[1/2][zF(z)]'$, *then* $\operatorname{Re}[f'(z)]>0$ *for* $|z|<(\sqrt{5}-1)/2$. *This result is sharp.*

PROOF. Let $F'(z)=P(z)$ where $P(0)=1$ and $\operatorname{Re}(P(z))>0$ for $z$ in $E$. We then have

$$2f'(z) = 2F'(z) + zF''(z) = 2P(z) + zP'(z).$$

Using again the fact that $|P'(z)| \leqq 2\operatorname{Re}[P(z)]/[1-|z|^2]$ for $z$ in $E$, we have

$$\begin{aligned} 2\operatorname{Re}[f'(z)] &\geqq 2\operatorname{Re}[P(z)] - |z||P'(z)| \\ &\geqq 2\operatorname{Re}[P(z)]\left[1-\frac{|z|}{1-|z|^2}\right] \\ &= 2\operatorname{Re}[P(z)]\left[\frac{1-|z|-|z|^2}{1-|z|^2}\right]. \end{aligned} \tag{14}$$

The right hand side of (14) is positive provided $|z|<(\sqrt{5}-1)/2$.

To see that the result is sharp, let $F(z)=-z-2\log(1-z)$. Then $f(z)=[1/2][2z^2/(1-z)-2\log(1-z)]$ and $f'(z)=(1+z-z^2)/(1-z)^2 =0$ for $z=(1-\sqrt{5})/2$.

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