LINEAR METHODS WHICH SUM SEQUENCES OF BOUNDED VARIATION

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A complex sequence $\{z_p\}$ is said to be of bounded variation provided $\sum |z_p - z_{p+1}| < \infty$. In this paper we show that a matrix which sums every sequence of bounded variation also sums a convergent sequence not of bounded variation (Theorem 1). Indeed, if M is a countable set of matrices, each of which sums every sequence of bounded variation, then there is a convergent sequence not of bounded variation which every matrix in M sums (Theorem 2). Our proofs are by direct construction. We are indebted to the referee for the following observation: Theorem 1 follows from a rather inaccessible result of Mazur-Orlicz-Zeller (see p. 125 of [4] and p. 256 of [5]) to the effect that the set of all convergent sequences which a matrix sums, as an FK space, has a separable dual space, while the space of sequences of bounded variation does not, since its maximal subspace of null sequences is equivalent to $\{z\colon \sum |z_i| < \infty\}$ whose dual is the set of all bounded sequences.

A basic tool in this study is the fact [1], [3] that a matrix (a_{pq}) sums every sequence of bounded variation if and only if

$$
(1)\qquad
\begin{aligned}
&\text{(i)}\quad \{a_{pn}\}_{p=1}^{\infty} \text{ converges}, \qquad n = 1, 2, 3, \cdots, \\
&\text{(ii)}\quad \left\{ \sum_{p=1}^{\infty} a_{np} \right\}_{n=1}^{\infty} \text{ converges, and} \\
&\text{(iii)}\quad \text{there exists } k \text{ such that} \left| \sum_{p=1}^{j} a_{np} \right| < k, \qquad n, j \geqq 1.
\end{aligned}
$$

We will denote the set of all convergent complex sequences by S_C and the set of all complex sequences of bounded variation by S_{BV}. Throughout, we will use the notation $x = \{x_p\}$, and, given a sequence y and an increasing sequence α of positive integers, we will use $F(y, \alpha)$ to denote the sequence z such that for each positive integer j, $z_p = y_j$, $\alpha_{j-1} < p \leqq \alpha_j$, $(\alpha_0 = 0)$.

LEMMA. *If $\sum a_p$ has bounded partial sums $\{s_p\}$, then there exists an increasing sequence α of positive integers such that if y is a null sequence and $F(y, \alpha) = x$, then $\sum x_p a_p$ converges.*

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PROOF. Let α be an increasing sequence of positive integers such that $\{s_{\alpha_p}\}\in S_{BV}$. Let y be a null sequence and $F(y, \alpha)=x$. Suppose $k>\alpha_1$ is an integer and t is the largest integer such that $\alpha_t<k$. Then, using summation by parts, we have

$$(2)\qquad \sum_{p=1}^{k} x_p a_p = \sum_{p=1}^{t-1} s_{\alpha_p}(x_{\alpha_p} - x_{\alpha_p+1}) + s_{\alpha_t}x_{\alpha_t} + \sum_{p=\alpha_t+1}^{k} x_p a_p.$$

By a well-known theorem of Hadamard [2], $\sum_{p=1}^{\infty} s_{\alpha_p}(x_{\alpha_p}-x_{\alpha_p+1})$ converges since $\sum_{p=1}^{\infty}(x_{\alpha_p}-x_{\alpha_p+1})=\sum_{p=1}^{\infty}(y_p-y_{p+1})$ converges and $\{s_{\alpha_p}\}_{p=1}^{\infty}$ converges absolutely, i.e., is of bounded variation. Thus from (2) we see that $\sum x_p a_p$ converges since $\{s_p\}$ is bounded, x and y are null sequences, and the last term on the right side of (2) is equal to $y_{t+1}(s_k-s_{\alpha_t})$. This completes the proof.

REMARK 1. Suppose $\sum a_p$ has bounded partial sums $\{s_p\}$ and α is an increasing sequence of positive integers such that $\{s_{\alpha_p}\}\in S_{BV}$. It is clear from the proof of the lemma that if α' is an increasing sequence of positive integers such that for some m, $\{\alpha_p'\}_{p=m}^{\infty}$ is a subsequence of α, y is a null sequence, and $F(y, \alpha')=x$, then $\sum x_p a_p$ converges.

THEOREM 1. *If A is a matrix which sums every sequence in S_{BV}, then there exists $x\in S_C-S_{BV}$ such that $Ax\in S_C$.*

PROOF. Suppose $A=(a_{pq})$ sums every sequence in S_{BV}. Using (i) of (1), we let $a_n=\lim_{p\to\infty} a_{pn}$, and using $\{s_p\}$ for the partial sums of $\sum a_p$, we note that (i) and (iii) of (1) can be used to show that $|s_p|\leqq k$ for all $p\geqq 1$. Let α be an increasing sequence of positive integers such that $\{s_{\alpha_p}\}\in S_{BV}$. Let t_1 be a positive integer such that if $q\geqq t_1$, then $|a_{qi}-a_i|<1/2^{i+1}$, $i=1, 2, \cdots, \alpha_1$. Let $r_1=\alpha_1$ and let r_2 be the first term of α following α_1 such that if j is a positive integer, then $|\sum_{q=r_2+j}^{\infty} a_{pq}|<1/2^2$, $p=1, 2, \cdots, t_1$. Let t_2 be an integer greater than t_1 such that if $q\geqq t_2$, then $|a_{qi}-a_i|<1/2^{i+2}$, $i=1, 2, \cdots, r_2$. Let r_3 be the first term of α following r_2 such that if j is a positive integer, then $|\sum_{q=r_3+j}^{\infty} a_{pq}|<1/2^3$, $p=1, 2, \cdots, t_2$. We continue the process to obtain sequences $\{r_p\}$ and $\{t_p\}$ such that $\{r_p\}$ is a subsequence of α, $\{t_p\}$ is an increasing sequence of positive integers, and if m if a positive integer, then for $q\geqq t_m$,

$$(3)\qquad |a_{qi} - a_i| < 1/2^{i+m}, \qquad i = 1, 2, \cdots, r_m,$$

and if j is a positive integer and $m>1$, then

$$(4)\qquad \left|\sum_{q=r_m+j}^{\infty} a_{pq}\right| < 1/2^m, \qquad p = 1, 2, \cdots, t_{m-1}.$$

Suppose y is a null sequence and let $F(y, r)=x$. By using (4), we can easily show that if n is a positive integer, then $\sum x_p a_{np}$ converges. Hence Ax is a complex sequence.

We next show that $Ax \in S_C$. By Remark 1 we note that $\sum x_p a_p$ converges. Let L be such that $|y_p|<L$, $p\geqq 1$. Suppose $\epsilon>0$. Let Q be a positive integer such that $1/2^Q<\epsilon$ and if $p>Q$, then $|x_p|<\epsilon$ and $|\sum_{j=p}^{\infty} x_j a_j|<\epsilon$. Let R be a positive integer such that if $t_p>R$, then $p>Q$. Let $N=Q+R$ and suppose $n>N$. If $t_q \leqq n < t_{q+1}$, then by use of (3) and (4), we have

$$\begin{aligned}\left|\sum_{p=1}^{\infty} x_p a_{np} - \sum_{p=1}^{\infty} x_p a_p\right| &\leqq \left|\sum_{p=1}^{r_q} x_p(a_{np}-a_p)\right| + \left|\sum_{p=r_q+1}^{\infty} x_p a_p\right| \\ &\quad + \left|\sum_{p=r_q+1}^{r_{q+1}} x_p a_{np}\right| + \left|\sum_{p=r_{q+1}+1}^{r_{q+2}} x_p a_{np}\right| \\ &\quad + \left|\sum_{p=r_{q+2}+1}^{\infty} x_p a_{np}\right| \\ &< 2L/2^{q+1} + \epsilon + 2k\epsilon + 2k\epsilon + 2L/2^{q+1} \\ &< (4L+1+4k)\epsilon.\end{aligned}$$

Thus $Ax \in S_C$, and we see that Ax converges to $\sum_{p=1}^{\infty} x_p a_p$. The theorem follows since we can take y to be a null sequence not of bounded variation and then $x \in S_C - S_{BV}$.

REMARK 2. The proof of Theorem 1 can be modified slightly to prove the following statement:

If A is a matrix which sums every sequence of bounded variation and α is an increasing sequence of positive integers, then there exists a subsequence β of α such that if γ is an increasing sequence of positive integers such that for some m, $\{\gamma_p\}_{p=m}^{\infty}$ is a subsequence of β, y is a null sequence, and $F(y, \gamma)=x$, then $Ax \in S_C$.

THEOREM 2. *If M is a countable set of matrices, each of which sums every sequence of bounded variation, then there exists a convergent sequence not of bounded variation which every matrix in M sums.*

PROOF. The proof will be for the case that M is infinite. Suppose $M=\{M_1, M_2, M_3, \cdots\}$. Let $\beta^{(1)}$ be an increasing sequence of positive integers which has the property with respect to M_1 which β has with respect to A in Remark 2. Let $\beta^{(2)}$ be a subsequence of $\beta^{(1)}$ such that $\beta^{(2)}$ has the property with respect to M_2 which β has with respect to A in Remark 2. We continue the process to obtain $\beta^{(1)}$, $\beta^{(2)}$, $\beta^{(3)}, \cdots$ such that if q is a positive integer, then $\beta^{(q+1)}$ is a subsequence of the increasing sequence $\beta^{(q)}$ of positive integers, and $\beta^{(q+1)}$

has the property with respect to M_{q+1} which β has with respect to A in Remark 2. Let $\beta_1^{(0)}=\beta_1^{(1)}$ and let $\beta_2^{(0)}$ be the first integer greater than $\beta_1^{(0)}$ such that $\beta_2^{(0)}$ is a term of $\beta^{(2)}$. Let $\beta_3^{(0)}$ be the first integer greater than $\beta_2^{(0)}$ such that $\beta_3^{(0)}$ is a term of $\beta^{(3)}$. Continue the process. Suppose y is a null sequence and $F(y, \beta^{(0)})=x$. Let n be a positive integer. Clearly there exists m such that $\{\beta_p^{(0)}\}_{p=m}^{\infty}$ is a subsequence of $\beta^{(n)}$. Thus by Remark 2, $M_n x \in S_C$. In particular, if y had been chosen as a null sequence not of bounded variation, then x would have been a convergent sequence not of bounded variation. Thus the theorem follows.

We now show that in Theorem 2 we cannot replace the set M with the set of *all* matrices which sum every sequence of bounded variation. To this end, suppose $\{b_p\} \in S_C - S_{BV}$. By a theorem of Hadamard [2], there exists $\{a_p\} \in S$ such that $\sum a_p$ converges and $\sum b_p a_p$ diverges. Let $A=(a_{pq})$ be defined as follows: $a_{pq}=a_q$ if $p \geqq q$, $a_{pq}=0$ if $p<q$. Clearly A satisfies (i), (ii), and (iii) of (1) and therefore sums every sequence of bounded variation, but A does not sum $\{b_p\}$.

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