ON THE INTEGRAL MODULI OF CONTINUITY IN $L_p$ ($1 < p < \infty$) OF FOURIER SERIES WITH MONOTONE COEFFICIENTS

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1. Introduction and results. Let $f(x)$ be of period $2\pi$ and integrable $L_p$ ($1 < p < \infty$). The integral moduli of continuity of first and second order of $f$ in $L_p$ are defined by

$$\omega_p(h; f) = \sup_{|t| \leq h} \|f(x + t) - f(x)\|_p$$

and

$$\omega_p^*(h; f) = \sup_{0 < t \leq h} \|f(x + t) + f(x - t) - 2f(x)\|_p$$

respectively, where $\| \cdot \|_p$ denotes the norm in $L_p$. The Lipschitz and Zygmund classes $\Lambda_p$ and $\Lambda_p^*$ are then defined by $\omega_p(h; f) = O(h)$ and $\omega_p^*(h; f) = O(h)$ respectively.

The problem of what can be said about the integral modulus of continuity (of first order) of the functions of the class $\Lambda_p^*$ ($1 < p < \infty$) was solved by A. Timan and M. Timan [6] for $p = 2$ and in the general case by Zygmund [7] in the following way:  

$$A_p h \log h \ll p_1$$

for $1 < p \leq 2$,

$$A_p h \log h \ll p_1^2$$

for $2 \leq p < \infty$.

Both estimates are best possible in general. Here we shall show that the second estimate can be improved for a special class of functions.

**Theorem 1.** If $f \in L_p$ ($1 < p < \infty$) has a cosine or sine Fourier series with monotone coefficients, then

$$f \in \Lambda_p^* \Rightarrow \omega_p(h; f) \leq A_p h \log h \ll p^{1/p}.$$  

The example of the function $f(x) = \sum_{n=1}^{\infty} n^{1/p-2} \cos nx$ ($1 < p < \infty$), which belongs to $\Lambda_p^*$ and whose integral modulus $\omega_p(h; f)$ is $> C_p h \log h \ll p^{1/p}$. Zygmund [7] shows that the estimate of Theorem 1 is the best possible.

Recently Aljančić and Tomic [1] proved that if the sequence $\{\mu_n\}$ satisfies $\mu_n \geq \mu_{n+1} \rightarrow 0$ and for a fixed $\check{p}$ ($1 < \check{p} < \infty$)

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$^1$ $A_p, B_p, \ldots$ denote constants which depend at most on $p$, but not necessarily always the same.
(1) \[ \sum_{r=1}^{n-1} \mu_r^{1-1/p} = O(n^{2-1/p} \mu_n) \quad \text{and} \quad \left\{ \sum_{r=n}^{\infty} \mu_r^{p-2/p} \right\}^{1/p} = O(n^{1-1/p} \mu_n), \]
then

(2) \[ \omega_p(n^{-1}; f) \leq A_p n^{1-1/p} \mu_n, \]
where \( f \) is the sum of either of the series

(3) \[ \sum_{n=1}^{\infty} \mu_n \cos nx \quad \text{or} \quad \sum_{n=1}^{\infty} \mu_n \sin nx. \]

We shall prove here a more complete result:

**Theorem 2.** Let \( \{\mu_n\} \) be a sequence which is monotonically decreasing to zero and such that for a fixed \( p \) \((1 < p < \infty)\)

(4) \[ \sum_{n=1}^{\infty} n^{p-2/p} \mu_n < \infty. \]

If \( f \) is the sum of either of the series (3), then

(5) \[ \omega_p(n^{-1}; f) \leq A_p n^{-1} \left\{ \sum_{r=1}^{n-1} \mu_r^{2p-2/p} \right\}^{1/p} + B_p \left\{ \sum_{r=n}^{\infty} \mu_r^{p-2/p} \right\}^{1/p}. \]

On account of A. Timan [5, p. 339]

\[ \left\{ \sum_{r=1}^{n-1} \mu_r^{2p-2/p} \right\}^{1/p} \leq A_p \sum_{r=1}^{n-1} \mu_r \]

the estimate (2) is included in that of (5). On the other hand, if \( \mu_n = n^{-\alpha} \) with \( \alpha > 1 - 1/p \), both (2) and (5) give the same estimate

\[ \omega_p(n^{-1}; f) = O(n^{1-1/p-\alpha}) = O(n^{1-1/p} \mu_n) \quad \text{when} \quad \alpha < 2 - 1/p, \]

but, for \( \alpha = 2 - 1/p \), (2) cannot be applied because of (1), whereas (5) gives

\[ \omega_p(n^{-1}; f) = O(n^{-1} \log^{1/p} n) = O(n^{1-1/p} \mu_n \log^{1/p} n). \]

As well as Theorem 1, the following theorem is partly based on a special case of Theorem 2.

**Theorem 3.** If \( \mu_n \geq \mu_{n+1} \to 0 \), then

(6) \[ \sum_{n=1}^{\infty} n^{2p-2} \mu_n < \infty \quad \text{for a fixed} \quad p \quad (1 < p < \infty) \]

\[^1\alpha > 1 - 1/p \] is necessary to guarantee the convergence of the series in (4).
is a necessary and sufficient condition that the sum $f$ of either of the series (3)

(i) belongs to $\Lambda_p$, or

(ii) is equivalent to an absolutely continuous function whose derivative belongs to $L_p$.

We remark that the results of Theorems 1–3 can be extended in an obvious manner to higher moduli and derivatives respectively. For example, for the modulus of order $k$, only the first term on the right side in (5) is to be replaced by

$$A_{p,k} = \left\{ \sum_{r=1}^{n-1} \nu^{(k+1)p-2} \frac{\mu_r}{\nu} \right\}^{1/p}.$$

2. Proof of Theorem 2. We note first that condition (4) is both necessary and sufficient that $f \in L_p$ [8, Chapter XII, Lemma 6.6]. We shall prove the theorem for the cosine series, the proof for the sine series being analogous.

On account of the symmetry of $f(x)$

$$\sup_{0 < |t| \leq h} \left\{ \int_{-\infty}^{\infty} |f(x + t) - f(x)|^p dx \right\}^{1/p}$$

$$= \sup_{0 < |t| \leq h} \left\{ \int_{0}^{\infty} |f(x - t) - f(x)|^p dx + \int_{0}^{\infty} |f(x + t) - f(x)|^p dx \right\}^{1/p},$$

the function of $t$ in the braces on the right side being pair. Hence, it suffices to evaluate

$$I = \left\{ \int_{0}^{\infty} |f(x \pm t) - f(x)|^p dx \right\}^{1/p} \quad \text{for } 0 \leq t \leq h.$$

Let $h = \pi/2n$. Owing to $(a+b)^{1/p} \leq a^{1/p} + b^{1/p}$ ($p > 1$), we have

$$I \leq \left\{ \int_{0}^{\pi/n} |f(x \pm t) - f(x)|^p dx \right\}^{1/p}$$

$$+ \left\{ \int_{\pi/n}^{\pi} |f(x \pm t) - f(x)|^p dx \right\}^{1/p} = I_1 + I_2.$$

By Minkowski's inequality

$$I_1 \leq 2 \left\{ \int_{0}^{\pi/n} \left| \sum_{r=1}^{n-1} \mu_r \sin \frac{\nu t}{2} \sin \nu(x \pm \frac{1}{2} t) \right|^p dx \right\}^{1/p}$$

$$+ \left\{ \int_{0}^{\pi/n} \left| \sum_{r=n}^{\infty} \mu_r [\cos \nu(x \pm t) - \cos \nu x] \right|^p dx \right\}^{1/p} = I_{11} + I_{12}.$$
As, by Hölder’s inequality,
\[ \sum_{\nu=1}^{n-1} \nu \mu_\nu \leq A_p n^{1/p} \left( \sum_{\nu=1}^{n-1} \frac{\nu^{2p-2}}{\mu_\nu} \right)^{1/p}, \]
we get
\[ (9) \quad I_{11} \leq t \left\{ \int_0^{\pi/n} \left( \sum_{\nu=1}^{n-1} \nu \mu_\nu \right)^p \, dx \right\}^{1/p} \leq A_p n^{-1} \left\{ \sum_{\nu=1}^{n-1} \nu^{2p-2} \mu_\nu \right\}^{1/p}. \]

For the latter of the integrals in (8) we find in virtue of \( t \leq \pi/2n \)
\[ I_{12} \leq \left\{ \int_{-\pi/2}^{\pi/2} \left| \sum_{\nu=1}^{\infty} \mu_\nu \cos \nu x \right|^p \, dx \right\}^{1/p} \]
\[ + \left\{ \int_0^{\pi/n} \left| \sum_{\nu=1}^{\infty} \mu_\nu \cos \nu x \right|^p \, dx \right\}^{1/p} \]
\[ \leq (2^{1/p} + 1) \left\{ \int_0^{\pi/2n} \left| \sum_{\nu=1}^{\infty} \mu_\nu \cos \nu x \right|^p \, dx \right\}^{1/p} \]
\[ \leq 3 \left\{ \sum_{m=1}^{\infty} \int_0^{\pi/2m} \left| \sum_{\nu=1}^{\infty} \mu_\nu \cos \nu x \right|^p \, dx \right\}^{1/p}. \]

As, for \( 3\pi/2(m+1) \leq x \leq 3\pi/2m \) \( (m = n, n+1, \ldots) \),
\[ \left| \sum_{\nu=1}^{\infty} \mu_\nu \cos \nu x \right| \leq \sum_{\nu=1}^{m} \mu_\nu + \pi x^{-1} \mu_{m+1} \leq \sum_{\nu=1}^{m} \mu_\nu + \frac{\pi}{3}(m+1)\mu_{m+1}, \]
we see that
\[ I_{12}^p \leq A_p \sum_{m=1}^{\infty} m^{-2} \left( \sum_{\nu=1}^{m} \mu_\nu \right)^p + B_p \sum_{m=1}^{\infty} m^{p-2} \mu_m^p. \]

Hardy’s inequality [3, Chapter IX, Miscellaneous theorems and examples 346]
\[ (10) \quad \sum_{m=1}^{\infty} m^{-2} \left( \sum_{\nu=1}^{m} C_\nu \right)^p \leq K_p \sum_{m=1}^{\infty} m^{p-2} C_m^p \quad (C_m \geq 0, p > 1), \]
with \( C_m = 0 \) for \( m < n \) and \( C_m = \mu_m \) for \( m \geq n \), shows that the first of these sums is majorized by the latter. Hence,
\[ (11) \quad I_{12} \leq B_p \left\{ \sum_{\nu=1}^{\infty} \mu_\nu \right\}^{1/p}. \]

If \( D_\nu(x) \) denotes the Dirichlet kernel, an Abel transformation
combined with Minkowski's inequality gives

\[
I_2 \leq \left\{ \int_{\pi/n}^{\pi} \left| \sum_{r=1}^{n} \Delta \mu_r \left[ D_r(x + t) - D_r(x) \right] \right|^p \, dx \right\}^{1/p} \\
+ \left\{ \int_{\pi/n}^{\pi} \left| \sum_{r=n+1}^{\infty} \Delta \mu_r \left[ D_r(x + t) - D_r(x) \right] \right|^p \, dx \right\}^{1/p} = I_{21} + I_{22}.
\]

By dividing the interval \((\pi/n, \pi)\) in subintervals \((\pi/(m+1), \pi/m)\) \((m = 1, \ldots, n-1)\) and applying \(D'_r(x) = O(\nu^2)\) for \(0 \leq x \leq \pi\) and \(D'_r(x) = O(x^{-3}) + O(\nu x^{-1}) = O(\nu x^{-1})\) for \(\pi/\nu \leq x \leq \pi\), one obtains in such a subinterval

\[
\sum_{r=1}^{n} \Delta \mu_r \left[ D_r(x + t) - D_r(x) \right] \\
\leq t \left( \sum_{r=1}^{m} + \sum_{r=m+1}^{n} \right) \Delta \mu_r \left| D'_r(x + \theta_r t) \right| \\
= O(t) \sum_{r=1}^{m} \nu^2 \Delta \mu_r + O(t) (x - t)^{-1} \sum_{r=m+1}^{n} \nu \Delta \mu_r, \quad (-1 < \theta_r < 1)
\]

because, on account of \(x \geq \pi/(m+1)\), the second estimate for \(D'_r(x)\) is applicable to every member in the latter sum. If we remember that by Abel transformation

\[
\sum_{r=1}^{m} \nu^2 \Delta \mu_r \leq 2 \sum_{r=1}^{m} \nu \mu_r, \quad \sum_{r=m+1}^{n} \nu \Delta \mu_r \leq \sum_{r=m+1}^{n} \nu \mu_r + m \mu_{m+1},
\]

and observe that, owing to \(t \leq \pi/2n\), the inequality \((x - t)^{-1} \leq 2x^{-1}\) \((x \geq 2t)\) may be applied in any of the mentioned subintervals, we get at last the following estimate:

\[
\left| \sum_{r=1}^{n} \Delta \mu_r \left[ D_r(x + t) - D_r(x) \right] \right| \\
= O(t) \sum_{r=1}^{m} \nu \mu_r + O(tm) \sum_{r=m+1}^{n} \mu_r + O(tm^2 \mu_m).
\]

Thus,

\[
I_{21}^p = \sum_{m=1}^{n-1} \int_{\pi/(m+1)}^{\pi/m} \left| \sum_{r=1}^{n} \Delta \mu_r \left[ D_r(x + t) - D_r(x) \right] \right|^p \, dx \\
= O(p) \left\{ \sum_{m=1}^{n-1} m^{-2} \left( \sum_{r=1}^{m} \nu \mu_r \right)^p + \sum_{m=1}^{n-1} m^{p-2} \left( \sum_{r=m+1}^{n} \mu_r \right)^p + \sum_{m=1}^{n-1} m^{2p-2} \mu_m \right\}.
\]
By Hardy’s inequality (10) with $C_m = m \mu_m$ for $m < n$ and $C_m = 0$ for $m \geq n$, the first of these sums is essentially majorized by the third. The same holds for the second sum according to another inequality of Hardy [3, ibid.]:

$$
\sum_{m=1}^{\infty} m^{-c} \left( \sum_{\nu=m}^{\infty} C_{\nu} \right)^p \leq K_p \sum_{m=1}^{\infty} m^{-c} C_m^p \quad (c < 1, C_m \geq 0, p > 1),
$$

if we choose $c = 2 - p$ and $C_m = \mu_{m+1}$ for $m < n$ and $C_m = 0$ for $m \geq n$. Hence,

(13) \quad I_{21} \leq A_p n^{-1} \left\{ \sum_{\nu=1}^{n-1} \nu^{2p-2} \frac{2p-2}{\mu_\nu} \right\}^{1/p}.

Lastly,

$$
I_{22} \leq 2 \left\{ \int_{\pi/2n}^{\pi+\pi/2n} \left| \sum_{\nu=n+1}^{\infty} \Delta_{\nu} D_{\nu}(x) \right|^p dx \right\}^{1/p} = O(\mu_n) \left\{ \int_{\pi/2n}^{\pi} x^{-p} dx \right\}^{1/p} = O(n^{1-1/p} \mu_n).
$$

As

$$
\left\{ \sum_{\nu=1}^{n-1} \nu^{2p-2} \frac{2p-2}{\mu_\nu} \right\}^{1/p} \geq \mu_{n-1} \left\{ \sum_{\nu=1}^{n-1} \nu^{2p-2} \right\}^{1/p} \geq C_p n^{2-1/p} \mu_n,
$$

one finds

(14) \quad I_{22} \leq A_p n^{-1} \left\{ \sum_{\nu=1}^{n-1} \nu^{2p-2} \frac{2p-2}{\mu_\nu} \right\}^{1/p}.

Collecting in (8) and (12) the estimates (9), (11), (13) and (14), from (7) follows Theorem 2.

3. Proof of Theorem 1. Recently, Konjuškov [4] called attention to the fact that if $f \in L_p (1 < p < \infty)$ has a cosine or sine series with monotone coefficients, then

(15) \quad \omega_p^*(n^{-1}; f) \geq C_p n^{1-1/p} \mu_n \quad (C_p > 0).

Konjuškov deduced (15) from his results about the relationship between the best trigonometric approximation of $f$ in $L_p$ and the Fourier coefficients of $f$. As (15), together with a special case of Theorem 2, is essentially in the proof of Theorem 1, we give here a direct

\footnote{He even supposed that for a fixed $\tau>0$, the sequence $n^{-\tau} \mu_n$ is only almost decreasing. His result is not limited to the modulus of second order only.}
The proof of (15). It is based on the following identity, easily verified:

\[
\frac{1}{4\pi} \int_{-\pi}^{\pi} [2f(x) - f(x + t) - f(x - t)] T_{m,n}(x) \, dx = \sum_{r=0}^{n} \mu_r \sin^2 \frac{\nu t}{2},
\]

where \( T_{m,n}(x) = \sum_{r=-m}^{n-m} \cos \nu x \) and \( f \) is a cosine series. If we set \( t = \pi/n \) in (16) and choose \( m = [n/2] \), then

\[
\sum_{r=0}^{n} \mu_r \sin^2 \frac{\nu \pi}{2n} \geq \frac{1}{n^2} \sum_{r=0}^{n} \nu^2 \mu_r \geq \frac{m^2}{n^2} \mu_n (n - m + 1) \geq C n \mu_n.
\]

On the other hand, in virtue of Hölder's inequality,

\[
\frac{1}{4\pi} \int_{-\pi}^{\pi} [2f(x) - f(x + \pi/n) - f(x - \pi/n)] T_{m,n}(x) \, dx \leq A_p n^{1/p} \left\{ \int_{-\pi}^{\pi} |f(x + \pi/n) + f(x - \pi/n) - 2f(x)|^p \, dx \right\}^{1/p}
\]

\[ \leq A_p n^{1/p} \omega_p^*(\pi/n; f), \]

because \( (1/q = 1 - 1/p) \)

\[
\int_{-\pi}^{\pi} |T_{m,n}(x)|^2 \, dx = 2 \left\{ \int_{0}^{\pi/n} O(n^q) \, dx + \int_{\pi/n}^{\pi} O(x^{-q}) \, dx \right\} = O(n^{q-1}).
\]

Hence, (15) follows.

The proof of Theorem 1 is now immediate. If \( \mu_n \) are the coefficients of \( f \) and \( \omega_p^*(n^{-1}; f) \leq A_p n^{-1} \), then, according to (15), \( \mu_n \leq B_p n^{-2+1/p} \) and one has only to apply Theorem 2 in this special case.

4. Proof of Theorem 3. On account of the well-known theorem of Hardy and Littlewood [2], which asserts the equivalence of (i) and (ii) in the general case, we have only to prove that (6)⇒(i) and that (ii)⇒(6).

(6)⇒(i). Because of \( s_n = \sum_{r=1}^{n} \nu^{2(p-2)} \mu_r < \infty \), an Abel transformation gives

\[
\sum_{r=0}^{\infty} \nu^{p-2} \mu_r = \sum_{r=0}^{\infty} \nu^{p} (s_r - s_{r-1}) = O(n^{-p}).
\]

Hence, the second term in (5) is also of order \( O(n^{-1}) \), i.e. \( f \in \Lambda_p \).

(ii)⇒(6). The proof proceeds, with necessary changes, along the same lines as the necessity part in the proof of Lemma 6.6 in [8, Chapter XII], and is adapted for the sine series.

Let
\[ F(x) = \int_0^x f(t) \, dt = \sum_{n=1}^{\infty} n^{-1} \mu_n \left( 1 - \cos nx \right). \]

Then, even simpler than in the proof of the cited lemma, \( F(\pi/n) \geq C \mu_n \). If we set

\[ G(x) = \int_0^x dt \int_0^t |f'(u)| \, du. \]

then \( F(x) \leq G(x) \). Hence, applying twice Hardy's inequality [8, Chapter I, p. 20], we get

\[
\sum_{n=2}^{\infty} n^{2p-2} \mu_n \leq A_p \sum_{n=2}^{\infty} n^{2p-2} G^p(\pi/n) \leq A_p \sum_{n=2}^{\infty} \int_{\pi/n}^{\pi/(n-1)} \left[ \frac{G(x)}{x} \right]^p x^{-p} \, dx \\
= A_p \int_0^{\pi} \left[ \frac{G(x)}{x} \right]^p x^{-p} \, dx \leq A_p \int_0^{\pi} \left( \int_0^x \left| f'(t) \right| \, dt \right)^p x^{-p} \, dx \\
\leq A_p \int_0^{\pi} \left| f'(x) \right|^p \, dx < \infty,
\]

which completes the proof of Theorem 3.

**References**


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