TOTAL NORMALITY AND THE HEREDITARY PROPERTY

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1. **Introduction.** In 1953 Dowker [4] introduced a new normality condition which he called *total normality*. He proved that every perfectly normal space is totally normal, and every totally normal space is completely normal. In this paper we prove further results about totally normal spaces. In particular, we show that if X is a totally normal space having a topological property P which satisfies certain axioms, then every subset of X has property P. Two main results are then as follows: (1) every totally normal paracompact space is hereditarily paracompact; (2) every totally normal collectionwise normal space is hereditarily collectionwise normal.

The reader is referred to E. Michael's papers [5], [6] for definitions concerning open covers, refinements, closure preserving collections, locally finite collections, and paracompactness, and to R. H. Bing's paper [1] for definitions concerning discrete collections and collectionwise normality. A topological space X is *completely normal* if every subset of X is normal. A topological space X is *hereditarily paracompact (hereditarily collectionwise normal)* if every subset of X is paracompact (collectionwise normal).

Let X be a normal space. X is said to be *perfectly normal* if every open subset of X is an F_σ in X, and X is said to be *totally normal* if every open subset U of X can be written as a locally finite (in U) collection of open F_σ subsets of X. All spaces in this paper are assumed to be Hausdorff.

2. **The axioms for P.** Let P be a topological property satisfying these axioms.

AXIOM 1. If X is any topological space such that every open subset of X has property P, then every subset of X has property P.

AXIOM 2. If X is any topological space having property P, and U is an F_σ subset of X, then U has property P.

AXIOM 3. If X is any topological space, and $\{V_\alpha : \alpha \text{ in } A\}$ is a locally finite open cover of X such that for all α in A, $\overline{V}_\alpha$ has property P, then X has property P.

We then have the following theorem.

THEOREM 1. *If X is a totally normal space having property P, then every subset of X has property P.*

PROOF. Let U be a subset of X; we want to show that U has property P. By Axiom 1, we may assume that U is open. Since X is totally normal, $U = \bigcup \{ U_\alpha : \alpha \text{ in } A \}$, where each U_α is an open F_σ in X, and the collection $\{ U_\alpha : \alpha \text{ in } A \}$ is locally finite in U. Since every totally normal space is completely normal [4, p. 276], the space U is normal. Hence the locally finite open cover $\{ U_\alpha : \alpha \text{ in } A \}$ of U has, by [2, p. 71], an open refinement $\{ V_\alpha : \alpha \text{ in } A \}$ such that for all α, F_α (F_α = closure of V_α in U) is contained in U_α. Now for all α in A, F_α is an F_σ in X, and so by Axiom 2, F_α has property P. It then follows by Axiom 3 that U has property P.

3. **Paracompactness.** Paracompactness is a topological property satisfying Axioms 1–3. Axiom 1 is due to Dieudonné [2, p. 68]; Axiom 2 is due to Michael [5, p. 835]; Axiom 3 is due to Michael [5, p. 831]. This proves Theorem 2. Note that Theorem 2 is a generalization of Dowker's result [3, p. 643] that every perfectly normal paracompact space is hereditarily paracompact.

THEOREM 2. *Every totally normal paracompact space is hereditarily paracompact.*

Dowker proved [4, p. 273] that every hereditarily paracompact space is totally normal. Combining this result with Theorem 2 gives the following theorem.

THEOREM 3. *Let X be a paracompact space. Then X is hereditarily paracompact if and only if X is totally normal.*

4. **Collectionwise normality.** Collectionwise normality is a topological property satisfying Axioms 1–3. Axiom 1 follows from Lemma 1 below; Axiom 2 is due to Šedivá [8, p. 53]; Axiom 3 follows from Lemma 2 below. This proves Theorem 4.

THEOREM 4. *Every totally normal collectionwise normal space is hereditarily collectionwise normal.*

LEMMA 1. *If every open subset of a topological space X is collectionwise normal, then every subset of X is collectionwise normal.*

PROOF. Let C be a subset of X, and let $\{ H_\alpha : \alpha \text{ in } A \}$ be a closed discrete collection in C. We want to find a mutually disjoint collection $\{ V_\alpha : \alpha \text{ in } A \}$ of sets open in C such that for all α in A, H_α is contained in V_α.

For each α in A, let

$$K_\alpha = \left(\bigcup_{\beta \neq \alpha} H_\beta \right)^{-}.$$

Then for each α in A, H_α and K_α form a pair of disjoint subsets of X. (This assertion follows immediately from the fact that $\{H_\alpha: \alpha$ in $A\}$ is a discrete collection in C.) Since K_α is a closed set, there is an open set U_α such that H_α is contained in U_α, and $U_\alpha \cap K_\alpha = \varnothing$. Note that for any $\beta \neq \alpha$, $U_\alpha \cap \overline{H}_\beta = \varnothing$.

Now let U be the open set $\bigcup\{U_\alpha: \alpha$ in $A\}$, and for each α in A, let $H_\alpha' = \overline{H}_\alpha \cap U$. Then the collection $\{H_\alpha': \alpha$ in $A\}$ has these properties:

(1) Each H_α' is closed in U.
(2) For each α in A, $H_\alpha \subset H_\alpha' \subset U_\alpha$.
(3) For $\alpha \neq \beta$, $U_\alpha \cap H_\beta' = \varnothing$.

From (1)–(3) it follows that the collection $\{H_\alpha': \alpha$ in $A\}$ is a closed discrete collection in U. (Each element of the collection is closed in U by (1); the collection is mutually disjoint since $H_\alpha' \subset U_\alpha$ and $U_\alpha \cap H_\beta' = \varnothing$, $\beta \neq \alpha$; the collection is locally finite in U, since given a point p in U, there is an α such that p is in U_α, and for $\beta \neq \alpha$, $U_\alpha \cap H_\beta' = \varnothing$.) Since U is collectionwise normal by hypothesis, there is a collection of mutually disjoint open sets $\{W_\alpha: \alpha$ in $A\}$ such that for all α, $H_\alpha' \subset W_\alpha$. Now let $V_\alpha = (W_\alpha \cap C)$; then $\{V_\alpha: \alpha$ in $A\}$ is a mutually disjoint collection of sets open in C such that for all α, H_α is contained in V_α.

LEMMA 2. *Let X be a topological space, and let $\{V_\alpha: \alpha$ in $A\}$ be a locally finite open cover of X such that for all α in A, $\overline{V}_\alpha$ is collectionwise normal. Then X is collectionwise normal.*

PROOF. Let $\{H_\sigma: \sigma$ in $B\}$ be a discrete collection of closed sets in X. We want to find a mutually disjoint collection of open sets $\{U_\sigma: \sigma$ in $B\}$ such that for all σ in B, H_σ is contained in U_σ.

For each α in A, the collection $\{\overline{V}_\alpha \cap H_\sigma: \sigma$ in $B\}$ is a discrete collection of closed sets in $\overline{V}_\alpha$. Since $\overline{V}_\alpha$ is collectionwise normal by hypothesis, there is a mutually disjoint collection $\{G_{\alpha,\sigma}: \sigma$ in $B\}$ of sets open in $\overline{V}_\alpha$ such that for each σ in B, $(\overline{V}_\alpha \cap H_\sigma)$ is contained in $G_{\alpha,\sigma}$. We may further assume that the collection $\{G_{\alpha,\sigma}: \sigma$ in $B\}$ is discrete in $\overline{V}_\alpha$ (and hence in X) by a result due to Nagami [7, p. 84]. Note that for any σ in B, $\overline{G}_{\alpha,\sigma}$ is contained in $\overline{V}_\alpha$, and that for $\sigma \neq \tau$, $\overline{G}_{\alpha,\sigma} \cap \overline{G}_{\alpha,\tau} = \varnothing$. Put $G = \{G_{\alpha,\sigma}: \alpha$ in A, σ in $B\}$; the collection G is closure preserving by a Lemma due to Michael [6, p. 827].

For each α in A and σ in B, let

$$H_{\alpha,\sigma} = \bigcup \left\{ \overline{G}_{\beta,\tau} : \beta \neq \alpha, \tau \neq \sigma \right\}.$$

Since G is closure preserving, each $H_{\alpha,\sigma}$ is closed. Now let $U_{\alpha,\sigma}$ $= (G_{\alpha,\sigma} - H_{\alpha,\sigma}) \cap V_{\alpha}$. The set $U_{\alpha,\sigma}$ is open in X. For, $(G_{\alpha,\sigma} - H_{\alpha,\sigma})$ is open in $\overline{V}_{\alpha}$, and so $(G_{\alpha,\sigma} - H_{\alpha,\sigma}) \cap V_{\alpha}$ is open in V_{α}, and therefore in X; i.e., $U_{\alpha,\sigma}$ is open in X.

Finally, for each σ in B, let

$$U_{\sigma} = \bigcup_{\alpha \text{ in } A} U_{\alpha,\sigma}.$$

Then the collection $\left\{ U_{\sigma} : \sigma \text{ in } B \right\}$ has these properties and thus is the desired mutually disjoint collection of open sets.

(1) Each U_{σ} is an open subset of X.
(2) For $\sigma \neq \tau$, $U_{\sigma} \cap U_{\tau} = \varnothing$.
(3) For all σ in B, $H_{\sigma} \subset U_{\sigma}$.

Properties (1) and (2) are clear. We shall prove (3). Let p be a point in H_{σ}. Since $\left\{ V_{\alpha} : \alpha \text{ in } A \right\}$ covers X, there is an α in A such that p is in V_{α}. Hence p is in $(\overline{V}_{\alpha} \cap H_{\sigma})$, and so p is in $G_{\alpha,\sigma}$. Suppose that for some $\tau \neq \sigma$, p is in $\overline{G}_{\beta,\tau}$. Then p is in $\overline{V}_{\beta}$, and so p is in $(\overline{V}_{\beta} \cap H_{\sigma})$ and therefore is in $G_{\beta,\sigma}$. But $\overline{G}_{\beta,\sigma} \cap \overline{G}_{\beta,\tau} = \varnothing$, a contradiction. Hence p is in $U_{\alpha,\sigma}$ and therefore is in U_{σ}.

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