A LAW OF THE ITERATED LOGARITHM FOR STABLE SUMMANDS

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Let x_n $(n=1, 2, 3, \cdots)$ be mutually independent random variables, identically distributed according to the symmetric stable distribution with exponent γ $(0<\gamma\leqq 2)$, i.e., $E[\exp(itx_n)]=\exp(-|t|^\gamma)$. Let $S_n=\sum_{k=1}^n x_k$. The classical "law of the iterated logarithm" (for the simplest exposition, see Feller [2, pp. 192–195]; see also [3] and [4]) tells us that for $\gamma=2$

$$P\left(\limsup_{n\to\infty}\frac{S_n}{\sqrt{(2n\log\log n)}}=1\right)=1.$$

That is, the variables $(1/\sqrt{n})S_n$ again satisfy $E[\exp(it(1/\sqrt{n})S_n)]=\exp(-|t|^2)$, and to achieve a finite lim sup they must be cut down additionally (and multiplicatively) by the factors $(2\log\log n)^{-1/2}$. For some reason the obvious corresponding statement for the case $\gamma<2$ does not seem to have been recorded, and it is the purpose of this note to do so.

For $0<\gamma<2$, the variables $n^{-\gamma^{-1}}S_n$ again satisfy $E[\exp(itn^{-\gamma^{-1}}S_n)]=\exp(-|t|^\gamma)$. Since the corresponding distribution function $F(x)$ has tail behavior $F(-x)+1-F(x)\sim(\text{const})|x|^{-\gamma}$ as $|x|\to\infty$, instead of exponential decrease as in the $\gamma=2$ case, we can expect the "cut down factors" to appear otherwise than as multipliers.

THEOREM. *For* $\gamma<2$

$$P\left(\limsup_{n\to\infty}|n^{-\gamma^{-1}}S_n|^{(\log\log n)^{-1}}=e^{\gamma^{-1}}\right)=1.$$

We sketch the proof. It suffices to show that for fixed $\epsilon>0$, and for almost every sample point, we have

$$(1)\qquad |n^{-\gamma^{-1}}S_n|>(\log n)^{(1+\epsilon)\gamma^{-1}}\qquad\text{finitely often}$$

and

$$(2)\qquad |n^{-\gamma^{-1}}S_n|>(\log n)^{(1-\epsilon)\gamma^{-1}}\qquad\text{infinitely often.}$$

Now the proof proceeds almost exactly as for the $\gamma=2$ case. Thus, to show (1), let A_n be the event that $|S_n|>n^{\gamma^{-1}}(\log n)^{(1+\epsilon)\gamma^{-1}}$. Pick

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$\beta > 1$, and for $r = 1, 2, 3, \cdots$, let n_r denote $[\beta^r]$, the largest integer in β^r. Let B_r denote the event that $|S_n| > (n_r)^{\gamma^{-1}} (\log n_r)^{(1+\epsilon)\gamma^{-1}}$ for some n with $n_r \leqq n < n_{r+1}$. Then $\limsup_{n\to\infty} A_n \subset \limsup_{r\to\infty} B_r$; and there exists a constant $c > 0$ independent of r such that for all r $P(B_r) \leqq cP(C_r)$, where C_r is the event that

$$(n_{r+1} - 1)^{-\gamma^{-1}} |S_{n_{r+1}-1}| > \left(\frac{n_r}{n_{r+1} - 1}\right)^{\gamma^{-1}} (\log n_r)^{(1+\epsilon)\gamma^{-1}}.$$

Since the distribution for $(n_{r+1}-1)^{-\gamma^{-1}} S_{n_{r+1}-1}$ has tail behavior $\sim(\text{const})|x|^{-\gamma}$ (cf. [**3**, pp. 181–182]), we conclude that for some finite constant $a > 0$, $P(C_r) \leqq ar^{-(1+\epsilon)}$, and $\sum_r P(B_r) < \infty$. Hence by the Borel Cantelli lemma, $P(\limsup_{n\to\infty} A_n) = P(\limsup_{r\to\infty} B_r) = 0$, and (1) holds.

To prove (2), set $D_r = S_{n_{r+1}} - S_{n_r}$. These are independent variables, and by the Borel Cantelli lemma again we find that for almost every sample point,

$$(n_{r+1} - n_r)^{-\gamma^{-1}} |D_r| \geqq (\log n_r)^{(1-\frac{1}{2}\epsilon)\gamma^{-1}}$$

for infinitely many r. Suppose that (2) does not hold on a set of positive probability. Then for almost every sample point in that set,

$$\begin{aligned} |n_{r+1}^{-\gamma^{-1}} S_{n_{r+1}}| &\geqq \left(1 - \frac{n_r}{n_{r+1}}\right)^{\gamma^{-1}} |D_r| - \left(\frac{n_r}{n_{r+1}}\right)^{\gamma^{-1}} |S_{n_r}| \\ &\geqq \left(1 - \frac{n_r}{n_{r+1}}\right)^{\gamma^{-1}} (\log n_r)^{(1-\frac{1}{2}\epsilon)\gamma^{-1}} - \left(\frac{n_r}{n_{r+1}}\right)^{\gamma^{-1}} (\log n_r)^{(1-\epsilon)\gamma-1} \end{aligned} \tag{3}$$

for infinitely many r. But for large r the last difference in (3) dominates $(\log n_{r+1})^{(1-\epsilon)\gamma^{-1}}$; so (2) does hold almost everywhere. For further details in this paraphrase of the classical case, we refer the reader to Feller [2], loc. cit.

REMARK. By stricting n to subsequences of the form $n_k = [\beta^{k^\delta}]$ for fixed $\beta > 1$ and $\delta > 1$, the proof shows that, with probability 1, every point in the interval $[1, e^{\gamma^{-1}}]$ is a limit point of the sequence $\{n^{-\gamma^{-1}} |S_n|^{(\log\log n)^{-1}}, n = 1, 2, 3, \cdots\}$. Now, at least for $1 < \gamma < 2$, 0 is also a limit point, as one can conclude from the general results of Chung and Fuchs (see [1, Theorem 4]). I do not know about the points in the interval (0, 1).

Added in proof. V. Strassen has pointed out to us that the above theorem follows simply from a result of A. Khinchine, Mat. Sb. **45** (1938); p. 582. However, the present version of the log log law holds also if the common d. f. F of the x_n lies in that part of the domain of *normal* attraction of a nonnormal stable d.f. G_γ $(0<\gamma<2)$ subject to conditions of the form

$$F(-x) = \frac{c_1}{x^\gamma} + \frac{d_1}{x^\delta} + r_1(x), \quad 1 - F(x) = \frac{c_2}{x^\gamma} + \frac{d_2}{x^\delta} + r_2(x),$$

where $r_i(x) = O(1/x^\epsilon)$ and $\gamma < \delta < \epsilon$ (and $r_1(x) + r_2(x)$ are monotone as $x \to \infty$ if $\gamma < 1$). For under these conditions, H. Cramer has shown (*On asymptotic expansions for sums of independent random variables with a limiting stable distribution*, Sankhya Ser. A **25** (1963), 12–24) that for the d.f. F_n of S_n (suitably shifted and scaled), $F_n(x) - G_\gamma(x) = O(1/n^\gamma)$ uniformly in x. Hence in the above proofs, we may replace tail estimates based on F_n by ones based on G_γ with an error of at most $O(1/n^\gamma)$. But on subsequences $n_j \subset [c^j]$, $c>1$, such errors will not affect the convergence or divergence of our series, and the proofs go through as before.

References

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