SMALL EIGENVALUES OF LARGE HANKEL MATRICES

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In this note we shall determine the asymptotic behavior as $N \to \infty$ of the smallest eigenvalue of the Hankel matrix

$$H_N = (c_{m+n}) \quad m, n = 0, \ldots, N.$$ 

It is assumed that the $c_n$ are the moments of a distribution function $\alpha(x)$ on the finite interval $[a, b]$,

$$c_n = \int_a^b x^n \, d\alpha(x),$$

where $w(x) = \alpha'(x)$ satisfies

$$\int_a^b \frac{\log w(x)}{(x - a)^{1/2}(b - x)^{1/2}} \, dx > -\infty.$$ 

We shall see that for the smallest eigenvalue $\lambda_N$ of $H_N$ there is an asymptotic formula of the form

$$\lambda_N \sim \rho N^{1/2} \sigma^{-2N}$$

where $\rho$ and $\sigma$ are constants which will be explicitly determined. In the case of the Hilbert matrix ($c_m = 1/(m+1)$) a partial result was obtained by Todd in [3]. (In certain exceptional cases the exponent $\frac{1}{2}$ must be replaced by $\frac{1}{4}$.) It will be found that $\sigma$ depends only on the interval $[a, b]$.

It will be assumed throughout that $a + b \geq 0$. This entails no loss of generality since the Hankel matrix corresponding to the distribution function $-\alpha(-x)$ on $[-b, -a]$ has exactly the same eigenvalues as $H_N$.

**Lemma 1.** Let $P_n(x)$ ($n = 0, 1, \ldots$) denote the orthogonal polynomials associated with $\alpha(x)$. Then $H_N^{-1}$ is similar to the matrix whose $m, n$ entry is

$$a_{m,n} = \frac{1}{2\pi} \int_0^{2\pi} P_m(e^{i\theta}) P_n(e^{i\theta})^* \, d\theta, \quad m, n = 0, \ldots, N.$$ 

**Proof.** Write $P_n(x) = \sum_{i=0}^n b_{n,i}x^i$. Then

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\[ \delta_{m,n} = \int_{a}^{b} P_m(x)P_n(x) \, d\alpha(x) = \sum_{i,j=0}^{N} b_{m,i}c_{i+j}b_{n,j} \]

and so if \( K_N \) denotes the matrix

\[
\begin{bmatrix}
  b_{0,0} & 0 & 0 & \cdots & 0 \\
  b_{1,0} & b_{1,1} & 0 & \cdots & 0 \\
  b_{2,0} & b_{2,1} & b_{2,2} & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  b_{N,0} & b_{N,1} & b_{N,2} & \cdots & b_{N,N}
\end{bmatrix}
\]

we have \( I = K_NH_NK_N^T \). Thus \( H_N^{-1} = K_N^T(K_NK_N^T)(K_N^T)^{-1} \). But the \( m,n \) entry of \( K_NK_N^T \) is

\[
\sum_{i=0}^{N} b_{m,i}b_{n,i} = \frac{1}{2\pi} \int_{0}^{2\pi} P_m(e^{i\theta})P_n(e^{i\theta})^* \, d\theta,
\]

which proves the lemma.

We shall be concerned now with the asymptotic behavior of \( a_{m,n} \) as \( m, n \to \infty \). This will turn out to be simple enough to enable us to deduce the asymptotic behavior of the largest eigenvalue of \( (a_{m,n}) \).

**Lemma 2.** We have, uniformly for \( z \) bounded away from the interval \([a, b]\),

\[ P_n(z) \sim (b - a)^{-1/2}n^{-1/2}\xi^nA(\xi), \]

where

\[ \xi = \frac{2}{b - a} z - \frac{b + a}{b - a} + \left( \frac{2}{b - a} z - \frac{b + a}{b - a} \right)^2 - 1 \right)^{1/2} \]

(the square root denoting that branch which is positive for large positive \( z \)), \( A(\xi) \) is analytic in \( |\xi| > 1 \) and

\[
\log | A(\rho e^{i\phi}) | = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \log \left[ \frac{b - a}{2} \cos t + \frac{b + a}{2} \right] | \sin t | \cdot \frac{\rho^2 - 1}{1 - 2\rho \cos(\phi - t) + \rho^2} \, dt.
\]

**Proof.** If \( a = -1, b = 1 \) this is Theorem 12.1.2 of [2] if \( \alpha(x) \) is absolutely continuous and is Theorem 9.3 of [1] for general \( \alpha \). The case of the interval \([a, b]\) may be reduced to this by a linear change of variable since if \( q_n(x) \) are the orthogonal polynomials associated
with the distribution function
\[ \alpha \left( \frac{b-a}{2} x + \frac{b+a}{2} \right) \]
on \([-1, 1]\) then
\[ P_n(x) = q_n \left( \frac{2}{b-a} x - \frac{b+a}{b-a} \right). \]

We omit the details.

In view of Lemma 2 we expect that the asymptotic behavior of \(a_{m,n}\) depends on the maximum of \(|\xi(z)|\) as \(z\) runs over the unit circle. The next lemma will describe this maximum. It is convenient at this point to distinguish three cases:

Case 1. \(a > -b/(1+2b)\).
Case 2. \(a = -b/(1+2b)\).
Case 3. \(a < -b/(1+2b)\).

**Lemma 3.** The maximum value of \(g(\theta) = |\xi(e^{i\theta})|\) is given by
\[
\sigma = \begin{cases} 
\left( \frac{b + a + 2}{b - a} \right)^{1/2} + \left[ \left( \frac{b + a + 2}{b - a} \right)^2 - 1 \right]^{1/2} & \text{Cases 1 and 2,} \\
\left( \frac{1}{|a|} \right)^{1/2} + \left( \frac{1}{|a|} \right)^{1/2} & \text{Cases 2 and 3.}
\end{cases}
\]
In Cases 1 and 2 the maximum occurs at \(\theta = \pi\) (and only there mod \(2\pi\)) and in Case 3 at \(\theta = \pm \theta_0\) (and only there mod \(2\pi\)) where
\[ \cos \theta_0 = \frac{b + a}{2ab}. \]
Moreover in Case 1 we have \(g''(\pi) \neq 0\), in Case 2 we have \(g''(\pi) = 0\) but \(g^{iv}(\pi) \neq 0\), and in Case 3 we have \(g''(\theta_0) \neq 0\).

The proof of the lemma is completely elementary and need not be reproduced here.

**Lemma 4.** There is a constant \(A\), depending only on the distribution function \(\alpha(x)\), such that for all \(m, n\)
\[
|a_{m,n}| \leq \begin{cases} 
\left( A(m + n + 1) \right)^{-1/2} \sigma^{m+n} & \text{Cases 1 and 3,} \\
\left( A(m + n + 1) \right)^{-1/4} \sigma^{m+n} & \text{Case 2.}
\end{cases}
\]

**Proof.** It follows from Lemma 2 that as long as the unit circle
does not intersect the interval \([a, b]\) we have

\[ |a_{m,n}| \leq \text{const} \int_0^{2\pi} g(\theta)^{m+n} \, d\theta \]

and the desired conclusions follow readily from Lemma 3 using standard techniques.

To show that the same estimates hold even if the unit circle does intersect \([a, b]\) let us assume that 1 belongs to the interval but \(-1\) does not. (The case in which they both belong to the interval is more complicated in only a trivial way.) We can write, for any \(\epsilon > 0\)

\[ |a_{m,n}| \leq \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \left| P_m(e^{i\theta})P_n(e^{i\theta}) \right| \, d\theta + \frac{1}{2\pi} \int_{\epsilon}^{2\pi-\epsilon} \left| P_m(e^{i\theta})P_n(e^{i\theta}) \right| \, d\theta. \]

Since the asymptotic formula of Lemma 2 holds uniformly for \(\epsilon \leq \theta \leq 2\pi - \epsilon\), the last integral will satisfy the estimate in the statement of the lemma. To estimate the first integral, denote by \(R_\epsilon\) the rectangle with vertices \(e^{\pm i\epsilon}, 1 \pm i \tan \epsilon\). This rectangle contains the arc of the unit circle given by \(|\theta| \leq \epsilon\). Since the polynomial \(P_m(z)P_n(z)\) has only real zeros (Theorem 3.3.1 of [2]) its maximum absolute value on \(R_\epsilon\) is attained on the horizontal sides of \(R_\epsilon\). On these sides we may apply the asymptotic formula of Lemma 2, and so

\[ \limsup_{m+n \to \infty} \max_{R_\epsilon} \left| P_m(z)P_n(z) \right|^{1/(m+n)} = g(\epsilon + O(\epsilon^2)). \]

Therefore we have as \(m+n \to \infty\)

\[ \int_{-\epsilon}^{\epsilon} \left| P_m(e^{i\theta})P_n(e^{i\theta}) \right| \, d\theta = O(t^{m+n}) \]

for any \(t > g(\epsilon + O(\epsilon^2))\). A little computation shows that \(g(2\epsilon) > g(\epsilon + O(\epsilon^2))\) if \(\epsilon\) is small enough. Thus

\[ \int_{-\epsilon}^{\epsilon} \left| P_m(e^{i\theta})P_n(e^{i\theta}) \right| \, d\theta = O(g(2\epsilon)^{m+n}). \]

But \(\sigma > g(2\epsilon)\), again for sufficiently small \(\epsilon\) (recall that \(g(\theta)\) does not attain its maximum \(\sigma\) at \(\theta = 0\)), and so certainly

\[ \int_{-\epsilon}^{\epsilon} \left| P_m(e^{i\theta})P_n(e^{i\theta}) \right| \, d\theta = o((m+n)^{-1/2}\sigma^{m+n}). \]

This completes the proof of the lemma.

The next lemma gives the asymptotic behavior of \(a_{m,n}\) as \(m, n \to \infty\). First some more notation. We write

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\[ \gamma = \begin{cases} 
\frac{|A(\zeta(-1))|^{\sigma^{1/2}}}{2^{1/2}\pi^{3/2}|g''(\pi)|^{1/2}(b-a)} & \text{Case 1,} \\
\frac{3^{1/4}\Gamma\left(\frac{1}{4}\right)|A(\zeta(-1))|^{\sigma^{1/4}}}{2^{9/4}\pi^{2}|g^{iv}(\pi)|^{1/4}(b-a)} & \text{Case 2,} \\
\frac{2^{1/2}|A(\zeta(e^{i\theta}))|^{\sigma^{1/2}}}{\pi^{3/2}|g''(\theta_0)|^{1/2}(b-a)} & \text{Case 3,} 
\end{cases} \]

where \(|A(\zeta)|\) is given in Lemma 2 and \(\theta_0\) in Lemma 3. We shall write, in Case 3,

\[ \text{sgn} \, \zeta(e^{i\theta_0}) = e^{i\Phi_0}. \]

(In Cases 1 and 2, \(\text{sgn} \, \zeta(-1) = -1.\))

**Lemma 5.** The following hold as \(m, n \to \infty\) with \(m-n\) bounded:

- \(a_{m,n} \sim \gamma(-1)^{m-n}(m+n)^{-1/2}e^{m+n}\) \hspace{1cm} \text{Case 1,}
- \(a_{m,n} \sim \gamma(-1)^{m-n}(m+n)^{-1/2}e^{m+n}\) \hspace{1cm} \text{Case 2,}
- \(a_{m,n} = \gamma \cos(m-n)\phi_0(m+n)^{-1/2}e^{m+n} + o((m+n)^{-1/2}e^{m+n})\) \hspace{1cm} \text{Case 3.}

**Proof.** Suppose the unit circle does not intersect \([a, b]\). (The case in which it does can be handled just as in the proof of Lemma 4.) Then by Lemma 2,

\[ a_{m,n} = \frac{1}{2\pi^2(b-a)} \int_0^{2\pi} \{g(\theta)^{m+n}[\text{sgn} \, \zeta(e^{i\theta})]^{m-n} |A(\zeta(e^{i\theta}))|^2 + o(g(\theta)^{m+n}) \} d\theta. \]

In Cases 1 and 2 the maximum of \(g(\theta)\) occurs at \(\theta = \pi\) (and nowhere else) and the result follows from Lemma 3 using standard techniques. In Case 3 the maximum occurs at \(\pm \theta_0\). Since

\[ \zeta(e^{-i\theta_0}) = (\zeta(e^{i\theta_0}))^* \quad |A(\zeta)| = |A(\zeta)| \]

the conclusion in this case also follows easily from Lemma 3.

**Theorem.** If \(\lambda_N\) is the smallest eigenvalue of \(H_N\), then as \(N \to \infty\),

- \(\lambda_N \sim \gamma^{-1}(\sigma^2 - 1)(2N)^{1/2}e^{-(N+1)}\) \hspace{1cm} \text{Case 1,}
- \(\lambda_N \sim \gamma^{-1}(\sigma^2 - 1)(2N)^{1/4}e^{-(N+1)}\) \hspace{1cm} \text{Case 2,}
- \(\lambda_N \sim 2\gamma^{-1} \left[\frac{1}{\sigma^2 - 1} + \left(\frac{1}{\sigma^4 - 2\sigma^2 \cos 2\Phi_0 + 1}\right)^{1/2}\right]^{-1}(2N)^{1/2}e^{-(N+1)}\) \hspace{1cm} \text{Case 3.}
PROOF. We shall consider in detail only Case 3; the others are easier. Let us write

\[
\begin{align*}
\beta_{m,n} &= \cos (m - n)\phi_0\sigma^{m+n}, \\
\gamma_{m,n} &= a_{m,n} - \gamma(2N)^{-1/2}\beta_{m,n}.
\end{align*}
\]

Fix \( N_0 \) and \( \epsilon \). It follows from Lemma 5 that if \( m \) and \( n \) are sufficiently large, but \( |m-n| \leq N_0 \), we shall have

\[
|a_{m,n} - \gamma \cos(m - n)\phi_0(m + n)^{-1/2}\sigma^{m+n}| \leq \epsilon(m + n)^{-1/2}\sigma^{m+n}.
\]

Therefore if both \( m \) and \( n \) exceed \( N - N_0 \) and \( N \) is sufficiently large we shall have

\[
|\gamma_{m,n}| = |a_{m,n} - \gamma \cos(m - n)\phi_0(2N)^{-1/2}\sigma^{m+n}|
\]

\[
\leq \epsilon(m + n)^{-1/2}\sigma^{m+n} + \gamma\sigma^{m+n}[(2N - 2N_0)^{1/2} - (2N)^{1/2}]
\]

\[
\leq \epsilon N^{-1/2}\sigma^{m+n}.
\]

It follows from Lemma 4 that for all \( m, n \)

\[
|\gamma_{m,n}| \leq A_1(m + n + 1)^{-1/2}\sigma^{m+n}
\]

where \( A_1 \) is a constant depending only on the distribution function \( \alpha(x) \). Denote by \( \mu_N \) the eigenvalue of largest absolute value of the matrix \((\gamma_{m,n})\) \((m, n = 0, \ldots, N)\). Then from (2) and (3) we obtain

\[
\begin{align*}
2\mu_N \leq & \sum_{m,n=0}^{N} \gamma_{m,n} \leq \epsilon N \sum_{m,n=0}^{N} \sigma^{2(m+n)} + 2A_1 \sum_{m=0}^{N-N_0} \sum_{n=0}^{N} \frac{\sigma^{2(m+n)}}{m + n + 1} \\
\leq & \frac{\epsilon^2\sigma^{4(N+1)}}{(\sigma^2 - 1)^2N} + A_2 \frac{\sigma^{2(2N-N_0)}}{2N - N_0},
\end{align*}
\]

where \( A_2 \) is another constant. If now \( N_0 \) is taken sufficiently large in comparison to \( \epsilon \), this will imply for sufficiently large \( N \)

\[
|\mu_N| \leq \frac{2\epsilon\sigma^{2(N+1)}}{(\sigma^2 - 1)N^{1/2}}.
\]

Now Lemma 1 implies that \( \lambda_N^{-1} \) is the largest eigenvalue of \((\beta_{m,n})\) \((m, n = 0, \ldots, N)\). It follows therefore from (1) and (4) that if \( \nu_N \) is the largest eigenvalue of \((\beta_{m,n})\) \((m, n = 0, \ldots, N)\), we have

\[
\begin{align*}
\gamma(2N)^{-1/2}\nu_N - \frac{2\epsilon\sigma^{2(N+1)}}{(\sigma^2 - 1)N^{1/2}} \leq & \lambda_N^{-1} \leq \gamma(2N)^{-1/2}\nu_N + \frac{2\epsilon\sigma^{2(N+1)}}{(\sigma^2 - 1)N^{1/2}}
\end{align*}
\]

for sufficiently large \( N \). Since the eigenvectors of \((\beta_{m,n})\) must be linear combinations \( \alpha \cos n\phi_0\sigma^n + \beta \sin n\phi_0\sigma^n \) it is easy to see that
\[ v_N \text{ is the largest eigenvalue of} \]
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} = \begin{bmatrix}
\sum_{0}^{N} \cos^2 n\phi_0 \sigma^{2n} & \sum_{0}^{N} \sin n\phi_0 \cos n\phi_0 \sigma^{2n} \\
\sum_{0}^{N} \sin n\phi_0 \cos n\phi_0 \sigma^{2n} & \sum_{0}^{N} \sin^2 n\phi_0 \sigma^{2n}
\end{bmatrix}.
\]

We find that as \( N \to \infty \)
\[
A = \frac{1}{2} \left[ \frac{1}{\sigma^2 - 1} + \frac{\sigma^2 \cos 2N\phi_0 - \cos 2(N + 1)\phi_0}{\sigma^4 - 2\sigma^2 \cos 2\phi_0 + 1} \right] \sigma^{2(N+1)} + O(1),
\]
\[
C = \frac{1}{2} \left[ \frac{1}{\sigma^2 - 1} - \frac{\sigma^2 \cos 2N\phi_0 - \cos 2(N + 1)\phi_0}{\sigma^4 - 2\sigma^2 \cos^2 \phi_0 + 1} \right] \sigma^{2(N+1)} + O(1),
\]
\[
B = \frac{1}{2} \frac{\sigma^2 \sin 2N\phi_0 - \sin 2(N + 1)\phi_0}{\sigma^4 - 2\sigma^2 \cos^2 \phi_0 + 1} \sigma^{2(N+1)} + O(1),
\]
and from these there follows easily
\[
(6) \quad v_N = \frac{1}{2} \left[ \frac{1}{\sigma^2 - 1} + \left( \frac{1}{\sigma^4 - 2\sigma^2 \cos 2\phi_0 + 1} \right)^{1/2} \right] \sigma^{2(N+1)} + O(1).
\]

The theorem follows from (6) and (5) if we observe that \( \epsilon \) was arbitrarily small.

We regret to announce that in the case of the Hilbert matrix
\[
\left( \frac{1}{m + n + 1} \right) \quad (m, n = 0, 1, \cdots, N)
\]
our result takes the form
\[
\lambda_N \sim 2^{9/8} \pi^{3/2}(73 - 48(2)^{1/2})^{-1} N^{1/2}(3 + 2(2)^{1/2})^{-2N-3/4} \quad (N \to \infty).
\]

REFERENCES


Cornell University and
University of Pennsylvania