UNITARY MATRICES AS BOUNDARY VALUES OF ANALYTIC FUNCTIONS

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Let \( Z \) be a matricial variable, and \( F(Z) \) an analytic matrix valued function. Under what circumstances is \( F(Z) \) unitary whenever \( Z \) is unitary? We can obtain a partial answer to this question in case \( F(Z) \) is entire.

We consider the function \( F(zU) \) of a single complex variable \( z \), where \( U \) is unitary. \( F(zU) \) is a function of the type already considered by Potapov [2], and we can adapt his results to obtain our conclusions.

A matrix valued function \( F(z) = [F_{ij}(z)] \) is simply an \( n \) by \( n \) square array of functions defined for all \( z \) in a domain \( D \), usually the unit disc. We will say that \( F(z) \) has a property such as analyticity, boundedness, etc., if the property is true of all the component functions.

Now let \( F(z) \) be a matrix valued function, analytic in the open unit disc \( |z| < 1 \) and continuous in the closed unit disc. We will suppose further that \( F(z) \) is unitary when \( |z| = 1 \). The collection \( \mathcal{S} \) of such functions forms a semigroup under (matrix) multiplication. Our first results will give the structure of this semigroup. It is convenient to set some notation:

(i) \( \| F(z) \| \) is the determinant of \( F(z) \).
(ii) \( F^T(z) \) is the transpose of \( F(z) \).
(iii) \( F^*(z) \) is the conjugate transpose of \( F(z) \).
(iv) \( f_\alpha(z) = |\alpha| / \alpha \cdot (\alpha - z)/(1 - \bar{\alpha}z), \quad 0 < |\alpha| < 1, \) and \( f_\theta(z) = z \), called Blaschke factors.
(v) \( F_\alpha(z) \) is the matrix

\[
\begin{bmatrix}
  f_\alpha(z) & 0 \\
  0 & I_{n-1}
\end{bmatrix}
\]

where \( I_{n-1} \) is the \( n - 1 \) dimensional identity matrix. (Note that \( F_\alpha(z) \) is unitary for \( |z| = 1 \).)

(vi) \( U \) will always denote a unitary matrix.

We start with the following simple lemma, a variant of the Schwarz reflection principle.

**Lemma 1.** Let \( F(z) \) be as described above. Suppose in addition that \( \| F(z) \| \) has no zeroes in \( |z| < 1 \). Then \( F(z) \) is a constant (necessarily a unitary matrix).

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Since \(\|F(z)\| \neq 0\) for \(|z| \leq 1\), \(F^{-1}(z)\) is analytic in \(|z| < 1\) and continuous in \(|z| \leq 1\). On the unit circle \(|z| = 1\) we have

\[ F(z)F^*(z) = I \]

which we write

\[ F(z)F^*(1/z) = I \]

or

\[ F(z) = (F^*(1/z))^{-1} \text{ for } |z| = 1. \]

But the expression of the right is analytic for \(|z| > 1\), and so gives an analytic continuation of \(F(z)\) into the extended plane. Consequently, \(F(z)\) is constant.

Returning to the general case, note that \(\|F(z)\|\) is analytic in \(|z| < 1\), continuous in \(|z| \leq 1\) and of absolute value one on \(|z| = 1\). Consequently, \(\|F(z)\|\) is a constant (of absolute value one) times a finite product of Blaschke factors,

\[ \|F(z)\| = e^{\sum_{v=1}^{k} f_{\alpha_v}(z)}. \]

We have \(\|F(\alpha_1)\| = 0\). Let \(u\) be a row vector of length one in the null-space of \(F(\alpha_1)\):

\[ uF(\alpha_1) = 0. \]

We construct a unitary matrix \(U_1\) whose first row is \(u\), so that \(U_1F(\alpha_1)\) has a first row identically zero. Thus \(U_1F(z)\) has a first row vanishing at \(z = \alpha_1\). Consequently, the function \(G(z) = (F_{\alpha_1}(z))^{-1}U_1F(z)\) is regular in \(|z| < 1\), and, as is immediately verified, belongs to \(S\). \(\|G(z)\|\) has one Blaschke factor less in its expansion; i.e.,

\[ \|G(z)\| = e^{\sum_{v=2}^{k} f_{\alpha_v}(z)}. \]

After a finite number of such reductions, we arrive at a function to which the result of Lemma 1 is applicable. We have proved:

**Theorem 1.** The semigroup \(S\) is generated by unitary matrices and matrices \(F_{\alpha}(z)\) \((0 \leq |\alpha| < 1)\).

If we had assumed at the start that \(F(z)\) was an entire function, then we would have \(\|F(z)\| = e^{\alpha z^k}\). Consequently:

**Corollary 1.** The sub-semigroup of \(S\) consisting of entire functions is generated by the unitary matrices and the single matrix \(F_0(z)\). Further-
more, if $F(z)$ is an entire function in $S$, then $\|F(z)\| = e^{\alpha z^k}$, and the entries in $F(z)$ are all polynomials of degree at most $k$.

Potapov has many interesting results on the uniform closure of the semigroup $S$.

We return now to the question in the initial paragraph of this paper. We have that

$$\|F(z U)\| = c(U)z^K$$

where, in fact, since the unitary group is connected, $K(U)$ must be constant $K$. Now from Corollary 1, we know that the entries in $F(z U)$ are all polynomials of degree at most $K$. Hence in the expansion of $F(z)$ in a power series, the homogeneous terms of weight greater than $K$ must vanish on the unitary group. This is known to insure their vanishing identically. Thus we have:

**Theorem 2.** Under the hypothesis described above, the entries in $F(z)$ are polynomials in the entries of $Z$.

Somewhat more can be proved easily. Our argument above will show that $\|F(z)\|$ is a homogeneous polynomial of weight $K$. For $Z$ unitary we have:

$$F(z) F^*(z) = I$$

which we write:

$$F(z) F^*(z^{*-1}) = I.$$

Since $F^*(z^{*-1})$ is analytic where it is defined, the above equation is true for all $Z$. Put $\text{ad } Z = \|z\| z^{-1}$. $\text{ad } Z$ is a polynomial in $Z$. By taking determinants above we obtain:

$$\|F(z)\| \|F^* (\frac{1}{\|z\|} \text{ad } Z^*)\| = I.$$

Using the homogeneity of $\|F(z)\|$ we have:

$$\|F(z)\| \|F^* (\text{ad } Z^*)\| = \|Z\|^K.$$

Since $\|Z\|$ is irreducible, $\|F(z)\| = c\|Z\|^L$.

In the special case that the dimension of $Z$ is that of $F(z)$, we note the following entire functions which are unitary when $Z$ is unitary:

(i) $Z$,

(ii) $Z^T$,

(iii) $\text{ad } Z$,

(iv) $\text{ad } Z^T$, 

It is possible that these together with the constant unitary matrices generate the whole class of such functions, but we have not been able to prove it.

References


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ON THE RADIUS OF UNIVALENCE OF CERTAIN ANALYTIC FUNCTIONS

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Let \( \mathcal{C} \) denote the class of functions \( f \) regular and univalent in \( E = \{ z | |z| < 1 \} \), which satisfy \( f(0) = 0 \) and \( f'(0) = 1 \) and which are close-to-convex in \( E \). Let \( \mathcal{K} \) and \( \mathcal{S}^* \) denote the subfamilies of \( \mathcal{C} \), made up of functions which are convex and starlike in \( E \), respectively. Recently, Libera \([2]\) has shown that if \( f \) is a member of \( \mathcal{K} \), \( \mathcal{S}^* \) or \( \mathcal{C} \), then the function \( F(z) = (2/z) \int_0^z f(t) dt \) is also a member of \( \mathcal{K} \), \( \mathcal{S}^* \) or \( \mathcal{C} \). It is the purpose of this paper to investigate the converse question. That is, if \( F \) is in \( \mathcal{S}^* \), what is the radius of starlikeness of the function \( f(z) = [1/2] [zF(z)]' \)? Similar questions are answered under the assumption that \( F \) is in \( \mathcal{K} \) or in \( \mathcal{C} \). Robinson \([5]\) has shown that if \( F \) is only assumed to be univalent in \( E \), then \( f \) is starlike for \( |z| < 0.38 \). He pointed out that it is probable that \( f \) is univalent for \( |z| < (1/2) \). We obtain this result under the added assumption that \( F \) is a member of \( \mathcal{K} \), \( \mathcal{S}^* \) or \( \mathcal{C} \).

The method of proof used in Theorem 1 has recently been employed by MacGregor \([4]\).

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