ON THE ZEROS OF SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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This note gives a proof of the following theorem:

In the differential equation

\[ y'' + p(x)y = 0, \]

where primes mean derivatives, suppose \( p(x) \) to be positive or zero, monotonic and concave (no point of an arc lies below its chord) in some closed interval \([a, b]\). If

\[ \int_a^b p(x) \, dx \geq (9/8)n^2\pi^2/(b - a), \]

where \( n \) is an integer, then every solution of (1) has at least \( n \) zeros in \([a, b]\). The number 9/8 cannot be replaced by a smaller one.

A theorem similar to this, but with more restrictive hypotheses, has been proved by Makai [5].

Related theorems have been proved by a number of authors, back as far as Liouville. More recent examples are given in the references.

The proof depends on three lemmas.

**Lemma 1.** If the equation

\[ y'' + q(x)y = 0, \]

where \( q(x) \) is continuous, has a solution with consecutive zeros at \( x = c \) and \( x = d \), and if

\[ \int_c^d q(x) \cos(2\pi(x - c)/(d - c)) \, dx \leq 0, \]

then

\[ \int_c^d q(x) \, dx \leq \pi^2/(d - c). \]

**Proof.** Let \( y(x) \) be the solution referred to, and let

\[ z(x) = (2/(d - c))^{1/2} \sin(\pi(x - c)/(d - c)), \]

so that \( \int_c^d z^2 \, dx = 1. \)

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\[ 0 \leq \int_{c}^{d} \frac{(z' y - z y')^2}{y^2} \, dx = \int_{c}^{d} \frac{(z' y - z y')(z/y)'}{y} \, dx \]

\[ = (z' y - z y')(z/y) \bigg|_{c}^{d} - \int_{c}^{d} (z/y)(z'' y - z y'') \, dx \]

\[ = \int_{c}^{d} \frac{\pi^2}{(d - c)^2} - q(x)^2 \, dx, \]

from (3). Since \( y(x) \) and \( z(x) \) have simple zeros at \( x = c \) and \( x = d \), their ratio has a limit at each end of the interval, so the integrated term vanishes. Then

\[ \frac{\pi^2}{(d - c)^2} \geq \int_{c}^{d} q(x)^2 \, dx \]

\[ = \left( \frac{1}{(d - c)} \right) \int_{c}^{d} q(x) \left[ 1 - \cos(2\pi(x - c)/(d - c)) \right] \, dx. \]

From this and (4), (5) follows.

**Corollary.** The number \( s \) is not less than \( d \), if \( s \) is determined by

\[ \int_{c}^{d} q(x) \, dx = \frac{\pi^2}{(s - c)}. \]

**Lemma 2.** If \( p(x) \) is concave, then

\[ \int_{c}^{d} p(x) \cos(2\pi(x - c)/(d - c)) \, dx \leq 0. \]

This lemma is due to E. Makai [4], pp. 370–371.

**Lemma 3.** Let \( \{x_i\} \) and \( \{x'_i\} \), \( i = 0, 1, 2, \ldots, n \), be two sets of numbers such that

(a) \( x_0 < x_1 < x_2 < \cdots < x_n, x'_0 < x'_1 < x'_2 < \cdots < x'_n \);

(b) \( x_1 - x_0 \geq x_2 - x_1 \geq x_3 - x_2 \geq \cdots \geq x_n - x_{n-1}, \)

and similarly for the \( \{x'_i\} \);

(c) \( x'_i \leq x_i, i = 1, 2, \ldots, n - 1; \)

(d) \( x_0 = x'_0, x_n = x'_n. \)

Then

\[ \sum_{i=1}^{n} \frac{1}{(x'_i - x'_i - 1)} \leq \sum_{i=1}^{n} \frac{1}{(x_i - x_{i-1})}. \]

**Proof.** The case \( n = 1 \) is trivial, and if \( n = 2 \) the proof is elementary.
To complete the proof by induction, let $S'_n$ and $S_n$ stand for the left and right members of (6) respectively. Then $S'_n \leq S_n$ must be shown to imply $S'_{n+1} \leq S_{n+1}$, with $n$ replaced by $n+1$ in (a), (b), (c) and (d). Let $g$ be the least of $x_i - x'_i$, and let $x'_i'' = x'_i + g$, $i = 1, 2, \cdots, n$, so that $x''_i - x''_{i-1} = x'_i - x'_i$. Let $S''_{n+1}$ be formed from $S'_{n+1}$ by substituting $x''_i$ for $x'_i$. Then $S''_{n+1} - S'_{n+1}$

\[ S''_{n+1} - S'_{n+1} = \frac{1}{(x_{n+1} - x'_{n'})} - \frac{1}{(x_{n+1} - x'_{n'})} + \frac{1}{(x'_i - x_0)} - \frac{1}{(x'_i - x_0)} \]

Now $x_{n+1} - x''_{n'} = x_{n+1} - x_n + (x'_n - x'_{n'}) - g \geq x_{n+1} - x_n > 0$, and $x_{n+1} - x''_{n'} \leq x_i - x'_{n'} \leq x'_i - x_0$ by (c) above; while $x'_i - x_0 \geq x'_i - x_0$. Hence the first denominator in the square bracket is not greater than the second, so that the bracket is positive or zero. Then $S''_{n+1} - S'_{n+1} \geq 0$.

But for at least one value of $i$, say $i = k$, $0 < k < n+1$, $x''_k' = x_k$. Then $S_{n+1}$ and $S'_{n+1}$ can each be broken into two sums, from $i = 1$ to $i = k$ and from $i = k+1$ to $i = n+1$ respectively. Each of these latter sums contains no more than $n$ terms and satisfies the hypotheses of the lemma. Hence the induction hypothesis applies to each, and their addition yields $S_{n+1} - S'_{n+1} \geq 0$. Addition of the previous inequality gives the lemma. (This proof, by E. Makai, Jr., was kindly sent to the author by the referee.)

**Proof of the Theorem.** Consider first that $p(x)$ has the special form

\[ r(x) = 2An^2\pi^2x, \]

where $A$ is a positive constant and $[a, b]$ is $[0, 1]$. Inequality (4) is satisfied, and it will appear later that $A$ can be $9/8$. Choose a solution $y(x)$ with $y(0) = 0$. Successive applications of Lemma 1 show that the succeeding zeros of $y(x)$ precede respectively the numbers $x_1, x_2, \cdots$, where the $x_i$ are determined by the equations

\[ \int_0^{x_1} r(x) \, dx = \pi^2/x_1 \quad \text{or} \quad An^2 x_1^3 = 1, \]

\[ \int_{x_1}^{x_2} r(x) \, dx = \pi^2/(x_2 - x_1) \quad \text{or} \quad An^2 (x_2^2 - x_1^2)(x_2 - x_1) = 1, \quad \text{etc}. \]

For $n = 1$, the theorem is true by Lemma 1. Assume it true for $n$. For $n+1$, the points $X_i$ corresponding to the $x_i$ are $X_i = (n/(n+1))^{2/3}x_i$, as is seen by substitution. The theorem will be true for $n+1$ if

\[ A(n+1)^2(X_{n+1}^2 - X_n^2)(X_{n+1} - X_n) = 1 \]
while $X_{n+1} \leq 1$, or if

$$A(n + 1)^2 \left\{1 - \left(\frac{n}{n + 1}\right)^{4/3} x_n^2\right\} \left\{1 - \left(\frac{n}{n + 1}\right)^{2/3} x_n\right\} \geq 1.$$ 

Since $x_n \leq 1$, $A$ can be chosen so that

$$A(n + 1)^2 \left\{1 - \left(\frac{n}{n + 1}\right)^{4/3}\right\} \left\{1 - \left(\frac{n}{n + 1}\right)^{2/3}\right\} \geq 1,$$

or

$$A\left\{(n + 1)^{2/3} - n^{2/3}\right\}^2\left\{(n + 1)^{2/3} + n^{2/3}\right\} \geq 1.$$

Let the part in braces be $f(n)$. Then $f(1) > 0.8914$, since $2^{2/3} > 1.587$. Then if $A = 9/8$, $Af(1) > 1.$

The behavior of $f(n)$ for large values of $n$ can be examined by treating $n$ as a continuous variable, increasing without limit. It can be shown by elementary arguments that $f'/f$ is negative and that the limit of $f(n)$ as $n$ increases is $8/9$. Hence if $A = 9/8$ the theorem is true for the function $r(x)$ considered.

Now let $p(x)$ be a function different from $r(x)$, and satisfying the hypotheses of the theorem. Since $\int_0^1 p(x) dx \geq (9/8)n^2 \pi^2$, $p(x)$ must be greater than $r(x)$ near $x = 0$ and less (perhaps) near $x = 1$. Suppose first that $p(x)$ is increasing. The equations (7), with $p(x)$ for $r(x)$, will determine numbers $x_1, x_2, \cdots$, which are not less than the respective zeros, by Lemma 1, and such that $x_1 \leq x_2 \leq \cdots$.

Now $x_i \leq x_i$, $i = 1, 2, \cdots, n$. For if not, let $x_i \leq x_i$, $i = 1, 2, \cdots, j - 1$, and suppose $x_j > x_j$. Then

$$(1/\pi^2) \int_0^x p(x) dx = 1/x_1 + 1/(x_2 - x_1) + \cdots + 1/(x_{j-1} - x_{j-1})$$

by Lemma 3, and this last sum is $(1/\pi^2)\int_0^x r(x) dx$. But if $F(x) = \int_0^x (p(t) - r(t)) dt$, $F'(x) = p(x) - r(x)$, which is zero at no more than one point between 0 and 1, by the concavity of $p(x)$. Hence $F(x)$ has at most one maximum, and is positive between 0 and 1. Since $p(x)$ is positive, $\int_0^x p(x) dx < \int_0^x p(x) dx < \int_0^x r(x) dx$ by (8), so $F(x_1) < 0$, a contradiction. Hence $x_j \leq x_j$, $j = 1, 2, \cdots, n$. The inequality above will apply directly if $n = 1$. Sturm's separation theorem shows that every other solution will have at least $n$ zeros. A linear change of variable from $[0, 1]$ to $[a, b]$ does not affect the argument. If $p(x)$ is decreasing, the same proof can be used from right to left.

That $9/8$ is the best possible constant can be shown thus: The
solution of \( y'' + xy = 0 \) that vanishes at the origin is \( y(x) = (x)^{1/2}J_{1/3}(2x^{3/2}/3) \), where \( J(\cdot) \) is the Bessel's function of the first kind. If \( t = 2x^{3/2}/3 \), the function \( Y(t) = (t)^{1/2}J_{1/3}(t) \) has zeros at points corresponding to those of \( y(x) \), and satisfies the equation

\[
\frac{d^2 Y}{dt^2} + \left( 1 + \frac{5}{36t^2} \right) Y = 0.
\]

The zeros \( t_n \) of \( Y(t) \) after the first will have the form

\[
t_n = n\pi + h + o(1/n),
\]

where \( h \) is some constant, since the interval between successive zeros approaches \( \pi \) as \( n \) becomes infinite. If a constant \( B < 9/8 \) could be used in (2), that inequality would show that the number \( v_n \), determined by \( \int_0^x xdx = Bn^2\pi^2/v_n \), was not less than \( x_n \), the \( n \)th positive zero of \( J_{1/3}(2x^{3/2}/3) \). This would imply

\[
B = \frac{v_n}{2n^2\pi^2} \geq \frac{x_n}{2n^2\pi^2} = \left( \frac{3t_n}{2} \right)^2/2n^2\pi^2 = \left( \frac{9}{8n^2\pi^2} \right)(n\pi + h + o(1/n))^2,
\]

which approaches 9/8. This completes the proof.

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References


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