A THEOREM ON LOCAL ISOMETRIES

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A mapping $\phi$ of a $G$-space $R$ (Busemann [1]) on itself is a locally isometric mapping if for each $x \in R$ there is a number $\eta_x > 0$ such that $\phi$ maps the spherical neighborhood $S(x, \eta_x)$ isometrically on $S(\phi(x), \eta_x)$. The problem we are concerned with is that of determining conditions on a $G$-space $R$ under which every locally isometric mapping of $R$ on itself is an isometry. Several such conditions have recently been given by Busemann [1, §27], [2], Szenthe [4], [5], and the author [3]. In this paper we are concerned with the more general of the conditions given by Szenthe [5].

For a fixed point $p \in R$, consider the collection $G(p)$ of all geodesic curves which begin and end at $p$, and which do not contain subarcs traversed more than once. For $h \in G(p)$, let $l(h)$ denote the length of $h$. Let $\lambda_i(p)$ and $\lambda_s(p)$ equal, respectively, inf $l(h)$ and sup $l(h)$ for all $h \in G(p)$. Put $\lambda_i(p) = \infty$ and $\lambda_s(p) = 0$ if $G(p)$ is empty. Let

$$\lambda_i = \inf_{p \in R} \lambda_i(p); \quad \lambda_s = \sup_{p \in R} \lambda_s(p).$$

Szenthe has proved [5] that if $\lambda_i > 0$ and $\lambda_s < \infty$, then every locally isometric mapping of $R$ on itself is an isometry.

Szenthe's condition provides a solution to a problem suggested by Busemann [1, p. 405] in that it applies to a cylinder with euclidean metric. It is known [1] that every locally isometric mapping of a compact $G$-space on itself is an isometry. Szenthe's condition, however, fails to hold in every compact space since, as he points out [5, p. 441], $\lambda_s = \infty$ on a torus with euclidean metric. Our purpose here is to present a condition more general than Szenthe's which holds in every compact space.

Let $\phi$ denote a locally isometric mapping of a $G$-space $R$ on itself. Let $\rho(p)$ be the supremum of those numbers $\rho$ such that if $x, y$ are in the spherical neighborhood $S(p, \rho)$, $x \neq y$, then there exists a point $z \neq y$ such that $xy + yz = xz$. We shall make use of the following properties of $\phi$ which are found in Busemann [1, §27].

(1) If $\phi$ is 1-1 then $\phi$ is an isometry.
(2) If $\phi(p_1) = \phi(p_2) = p$, $p_1 \neq p_2$, then $p_1 p_2 \geq 2\rho(p)$.
(3) The number of points of $R$ which lie over a given point of $R$ is countable, and is the same for different points of $R$.

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The following fact is an easy consequence of propositions (27.4) and (27.11) of Busemann [1].

(4) If \(a, b \in \mathbb{R}\) are given, and if \(\phi(\alpha) = a\), then there is a point \(\beta \in \mathbb{R}\) such that \(\phi(\beta) = b\) and \(\alpha \beta = ab\).

**Definition.** A geodesic curve \(h \in G(p)\) is called a circular loop at \(p\) provided there exists a \(z\) such that

\[
h = T_1(p, z) \cup T_2(p, z)
\]

where \(T_1(p, z)\) and \(T_2(p, z)\) denote (necessarily) distinct metric segments with endpoints \(p\) and \(z\).

Let \(Q(p)\) denote the collection of all circular loops at \(p\). For \(h \in Q(p)\), let \(l(h)\) denote the length of \(h\). Further, let

\[
l_i(p) = \inf \{l(h) : h \in Q(p)\},
\]

\[
l_s(p) = \sup \{l(h) : h \in Q(p)\},
\]

If \(Q(p) = \emptyset\), take \(l_i(p) = \infty\) and \(l_s(p) = 0\). Then let

\[
l_i = \inf \{l_i(p) : p \in \mathbb{R}\},
\]

\[
l_s = \sup \{l_s(p) : p \in \mathbb{R}\}.
\]

It will be helpful to establish the following lemma before turning to the theorem.

**Lemma.** Let \(\phi\) be a locally isometric mapping of \(\mathbb{R}\) on itself, and suppose that \(\phi(p_1) = p\). Let

\[
W = \{x \in \mathbb{R} : \phi(x) = p, x \neq p_1\}.
\]

If \(W \neq \emptyset\) then there is at least one point \(p_2 \in W\) such that \(p_1p_2 = \inf \{p_1x : x \in W\}\). Further, if \(T(p_1, p_2)\) is a segment joining \(p_1\) and \(p_2\), then \(\phi(T(p_1, p_2)) \in Q(p)\).

**Proof.** The first part of the conclusion follows easily. If \(x, y \in W\), \(x \neq y\), then \(xy \geq 2p(p)\) by (2). Hence by finite compactness of \(\mathbb{R}\), the set of points \(x \in W\) such that \(p_1x\) is less than a given number is finite.

We now prove that \(\phi(T(p_1, p_2)) \in Q(p)\). Let \(z_1\) denote the midpoint of \(T(p_1, p_2)\). By (4), there is a point \(p_3\) such that \(\phi(p_3) = p\) and \(z_1p_3 = zp\), where \(z = \phi(z_1)\). If \(p_3 = p_1\) then \(pz = p_1z_1 = (1/2)p_1p_2\). Otherwise

\[
p_1p_2 = p_1z_1 + z_1p_2 \geq p_1z_1 + zp = p_1z_1 + z_1p_3 \geq p_1p_3 \geq p_1p_2,
\]

and thus in either case \(pz = p_2z_1 = (1/2)p_1p_2\).

Let \(T(p_1, z_1)\) and \(T(z_1, p_2)\) be subsegments of \(T(p_1, p_2)\). Then \(\phi(T(p_1, z_1))\) and \(\phi(T(z_1, p_2))\) are segments since each is a geodesic arc whose length is equal to the distance between its endpoints. Therefore,
\[ \phi(T(p_1, p_2)) = \phi(T(z_1, z_2)) \cup \phi(T(z_1, p_2)) \subseteq Q(p). \]

**Theorem.** If \( R \) is a G-space for which \( l_t > 0 \) and \( l_\infty < \infty \), then every locally isometric mapping of \( R \) on itself is an isometry.

**Proof.** Let \( \phi \) be a locally isometric mapping of \( R \) on itself and suppose that \( \phi \) is not an isometry. Let \( p \in R \). We define the sequence \( \{ p_n \} \) as follows. By (1) \( \phi \) is not 1-1, so by the lemma, for each positive integer \( n \) there is a point \( p_n \) such that \( \phi(p_n) = p_{n+1} \) and

\[ p_n \phi^n(p) = \inf \{ x \phi^n(p) : \phi(x) = \phi^{n+1}(p), x \neq \phi^n(p) \}. \]

Using (4) and the fact that \( \phi^n \) is also a local isometry, let \( p_n \) be a point such that \( \phi^n(p_n) = p_i \) and \( p_p = \phi^n(p) p_i \).

By the lemma \( \phi \) maps \( T(\phi^n(p), p_i) \) into an element of \( Q(\phi^{n+1}(p)) \), which has length \( \phi^n(p) \phi_i \), and hence \( p_p \leq l_\infty \). Further, \( p_m \neq p_n \) if \( m \neq n \) since, assuming \( n < m \), \( \phi^n(p_m) = p_i \) while \( \phi^m(p_n) = \phi^m(p) \neq p_i \). Therefore, since \( R \) is finitely compact, there are integers \( k, l \) such that \( k_p l_i \) where \( k \neq l \). Assume \( k < l \). Then \( \phi^i(p_k) = p_i \) and \( \phi^l(p_k) = \phi^l(p) \). Thus \( \phi^i(p) = \phi^l(p) \). Since \( T(\phi^i(p), p_i) \) is mapped by \( \phi \) into an element of \( Q(\phi^{l+1}(p)) \) which has length \( \phi^i(p) \phi^l(p) \), we have a contradiction.

That our theorem is more general than Szenthe's is evident; the collection \( Q(p) \) is contained in the collection \( G(p) \). It is easily seen that our condition also holds in all compact spaces. An element of \( Q(p) \) is composed of two distinct segments \( T_1(p, z) \) and \( T_2(p, z) \). In a compact space the lengths of such segments are bounded above by the diameter of the space and bounded below by \( \inf \{ \rho(p) : p \in R \} \) which is positive (cf. [1, p. 39]).

**References**


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