ON FUNCTIONS THAT COMMUTE
WITH FULL FUNCTIONS

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A continuous function \( f \) mapping the unit interval \([0, 1] = I\) onto itself is said to be full if \( I \) can be partitioned into a finite number of subintervals \( J_i \) such that \( f \) maps each \( J_i \) homeomorphically onto \( I \). The number of subintervals \( J_i \) will be called the number of branches of \( f \). H. Cohen [2] showed that two full functions which commute (under composition) must have a common fixed point. In [1] Baxter and Joichi investigated the question of which continuous functions commute with full functions. The author conducted a similar investigation independently, and in this note we give some extensions of the results in [1]. Henceforth, it will be assumed that all functions considered are continuous.

Following Baxter and Joichi we define a hat function to be a piecewise linear full function whose derivative has constant absolute value. A full function \( f \) will be called regular if there is a homeomorphism \( \phi \) of \( I \) onto \( I \) such that \( \hat{f} = \phi f \phi^{-1} \) is a hat function. Baxter and Joichi show that if \( g \) commutes with a hat function \( f \) with at least two branches, then either \( g \) is a hat function or \( g \) is constantly equal to a fixed point of \( f \). From this it follows that if \( g \) commutes with a regular full function \( f \) having two or more branches then \( g \) is either full and regular or it is constantly equal to a fixed point of \( f \).

An example is given in [1] of a full function with two branches which commutes with a nonconstant, nonfull function. This shows that the above result does not hold for irregular full functions. However, the following generalization is valid.

**Theorem 1.** Let \( f \) be a full function with \( n \geq 2 \) branches. There is a continuous monotone increasing function \( \phi \) mapping \( I \) onto \( I \) and a hat function \( \hat{f} \) with \( n \) branches such that \( \phi f = \hat{f} \phi \). If \( g \) commutes with \( f \), there is a continuous function \( \bar{g} \) such that \( \phi g = \bar{g} \phi \) and \( \bar{g} \) commutes with \( \hat{f} \). Furthermore, \( \phi \) is a homeomorphism if and only if \( f \) is regular.

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Combining the results of Baxter and Joichi with Cohen's theorem, we see that if \( g \) commutes with a regular full function \( f \), then \( f \) and \( g \) have a common fixed point. Using Theorem 1 we obtain

**Theorem 2.** Let \( f \) be a full function. If \( g \) commutes with \( f \), then \( f \) and \( g \) have a common fixed point.

**Proof of Theorem 1.** For each positive integer \( k \), let
\[
0 = t(k, 0) < t(k, 1) < \cdots < t(k, n^k) = 1
\]
be the points where \( f^k \), the \( k \)th iterate of \( f \), assumes the values 0 and 1. Define \( \phi \) by
\[
\phi(x) = \sup \left\{ \frac{i}{n^k} \mid k > 0, \ 0 \leq i \leq n^k, \ t(k, i) \leq x \right\}.
\]
Clearly \( \phi \) is monotone increasing, \( \phi(0) = 0 \) and \( \phi(1) = 1 \). Since \( t(k, i) = t(k+1, ni) \), \( \phi(t(k, i)) = i/n^k \). Hence, \( \phi(I) \) is dense in \( I \). But \( \phi \) is monotone so this implies that \( \phi \) is continuous.

Let
\[
T^k = \{ t(k, i) \mid 0 \leq i \leq n^k \}
\]
and let \( T = \bigcup_k T^k \). Then \( T \) is the set of points in \( I \) which are mapped into \( \{0, 1\} \) by some iterate of \( f \). Now \( f \) is onto and \( f(\{0, 1\}) \subset \{0, 1\} \) so \( f(T) = f^{-1}(T) = T \). Furthermore, if \( 0 \leq x \leq y \leq 1 \), then a necessary and sufficient condition for \( \phi(x) \) to be equal to \( \phi(y) \) is that the interval \( [x, y] \) contain at most one point from the set \( T \).

Define \( \tilde{f} \) by
\[
\tilde{f}(x) = \phi(f^{-1}(x)).
\]
To show that \( \tilde{f} \) is well defined we must show that \( \phi(x) = \phi(y) \) implies that \( \phi(f(x)) = \phi(f(y)) \). Suppose not. Then there are \( x, y \in I \) such that at most one point of \( T \) is between \( x \) and \( y \) and two points of \( T \) are between \( f(x) \) and \( f(y) \). But these two points are the images under \( f \) of a pair of points between \( x \) and \( y \). This is a contradiction since \( f^{-1}(T) = T \).

We clearly have \( \tilde{f} \phi = \phi f \). Let \( Q \) be a closed subset of \( I \). Then \( \tilde{f}^{-1}(Q) = \phi(f^{-1} \phi^{-1}(Q)) \). Now \( f^{-1} \phi^{-1}(Q) \) is closed and hence compact by the continuity of \( f \) and \( \phi \). Therefore, \( \phi(f^{-1} \phi^{-1}(Q)) \) is compact and hence closed so \( \tilde{f} \) is continuous.

To show that \( \tilde{f} \) is a hat function with \( n \) branches, it is sufficient to show that, for each \( k > 0 \) and each \( i \), \( 0 \leq i < n \), \( \tilde{f} \) maps the points \( i/n+j/n^{k+1}, \ 0 \leq j \leq n^k \), monotonically onto the points \( l/n^k, \ 0 \leq l \leq n^k \).

This follows from
\[
\begin{align*}
\mathcal{J} \left( \frac{i}{n} + \frac{j}{n^{k+1}} \right) &= \mathcal{J} \phi(t(k + 1, n^k i + j)) = \phi f(t(k + 1, n^k i + j)) \\
&= \begin{cases} \\
\phi(t(k, j)) = \frac{j}{n^k} & \text{if } f \text{ is increasing on } [t(1, i), t(1, i + 1)], \\
\phi(t(k, n^k - j)) = \frac{n^k - j}{n^k} & \text{if } f \text{ is decreasing on } [t(1, i), t(1, i + 1)]. \\
\end{cases}
\end{align*}
\]

Now suppose that \( g \) commutes with \( f \). By the above argument, to show the existence of a continuous \( \mathcal{J} \) satisfying \( \phi g = \mathcal{J} \phi \) it suffices to show that \( \phi(x) = \phi(y) \) implies \( \phi g(x) = \phi g(y) \).

Suppose not. Then for some \( x, y \in I \) with \( x < y \), \( \phi(x) = \phi(y) \) and \( \phi g(x) \neq \phi g(y) \). If there is a point \( t \in T \) with \( x < t < y \), it is the only element of \( T \) in \([x, y]\). Now \( \phi(x) = \phi(t) = \phi(y) \) but either \( \phi g(x) \neq \phi g(t) \) or \( \phi g(t) \neq \phi g(y) \). Replacing \([x, y]\) by either \([x, t]\) or \([t, y]\), we may assume that there is no element \( t \in T \) with \( x < t < y \). Hence \( f^k \) is monotone on \([x, y]\) for every \( k \).

Since \( \phi g(x) \neq \phi g(y) \), for \( k \) sufficiently large there are an arbitrarily large number of consecutive points from \( T \) between \( g(x) \) and \( g(y) \). Hence, for any \( r \) there is a \( k \) and a monotone sequence \( x_1, \ldots, x_r \) of points between \( g(x) \) and \( g(y) \) such that \( f^k \) alternately assumes the values 0 and 1 on this sequence. The sequence \( x_1, \ldots, x_r \) is the image under \( g \) of a monotone sequence \( y_1, \ldots, y_r \) of points in \([x, y]\). To see this, let \( a, b \in I \). Now \( g([a, b]) \) is a closed interval containing \( g(a) \) and \( g(b) \). Hence, if \( c \) is between \( g(a) \) and \( g(b) \), then \( c \) is the image under \( g \) of a point between \( a \) and \( b \). Since \( x_1 \) and \( x_r \) are between \( g(x) \) and \( g(y) \), there are points \( y_1 \) and \( y_r \) between \( x \) and \( y \) with \( g(y_1) = x_1 \) and \( g(y_r) = x_r \). Now \( x_2 \) is between \( x_1 = g(y_1) \) and \( x_r = g(y_r) \), so there is a point \( y_2 \) between \( y_1 \) and \( y_r \) with \( g(y_2) = x_2 \). Continuing in this fashion, we construct a sequence of points \( y_1, \ldots, y_r \) in \([x, y]\) with \( g(y_i) = x_i \) and \( y_i \) between \( y_{i-1} \) and \( y_r \) for \( 1 < i < r \). This is the required monotone sequence.

The function \( f^k g = g f^k \) alternately assumes the values 0 and 1 on the sequence \( y_1, \ldots, y_r \). Since \( f^k \) is monotone on \([x, y]\), the sequence \( f^k(y_1), \ldots, f^k(y_r) \) is monotone and \( g \) alternately assumes the values 0 and 1 on this sequence. But \( r \) may be arbitrarily large, so this contradicts the continuity of \( g \).

Since \( \phi \) is onto, to show that \( \mathcal{J} g \mathcal{J} = \mathcal{J} g \phi \) it suffices to show that \( \mathcal{J} g \phi = \mathcal{J} g \phi \). This follows from the relations \( \mathcal{J} \phi = \phi f \), \( \mathcal{J} g \phi = \phi g \) and \( f g = g f \).

To see the final remark, observe that if \( \phi \) is a homeomorphism then
\(\phi f \phi^{-1} = f\) so \(f\) is regular. Conversely, if \(f\) is regular then the set \(T\) is dense in \([0, 1]\). This implies that \(\phi\) is 1-1 and hence a homeomorphism.

Before proceeding to the proof of Theorem 2, we need a

**Lemma.** Let \(f\) and \(g\) be continuous functions from \([a, b]\) to \([a, b]\) which commute. If \(f\) is monotone then \(f\) and \(g\) have a common fixed point.

**Proof.** If \(f\) is decreasing it has a unique fixed point \(x_0\). Now \(fg(x_0) = gf(x_0) = g(x_0)\) so \(g(x_0) = x_0\). If \(f\) is increasing, let \(x_0\) be a fixed point of \(g\). The sequence \(\{x_n\}\) defined by \(x_n = f(x_{n-1})\) for \(n > 0\) is monotone, so it has a limit which is a common fixed point of \(f\) and \(g\).

**Proof of Theorem 2.** If \(f\) has only one branch, then \(f\) is monotone and the conclusion follows from the lemma. If \(f\) has more than one branch then the hypotheses of Theorem 1 are satisfied. By [1] and [2], \(\tilde{f}\) and \(\tilde{g}\) have a common fixed point \(x_0\). Let \([a, b] = \phi^{-1}(x_0)\). Then \(\phi f([a, b]) = \tilde{f}(\phi([a, b])) = \tilde{f}(x_0) = x_0\), so \(f([a, b]) \subseteq [a, b]\). Similarly, \(g([a, b]) \subseteq [a, b]\).

Suppose \(f\) is not monotone on \([a, b]\). Then there is a \(t \in T^1\) with \(a < t < b\). Now \(f(t) \in [a, b]\) and \(f(t) \not\in T\), but \([a, b]\) contains at most one point from \(T\), so \(f(t) = t\). However, \(f(t)\) is either 0 or 1 and neither of these points are in the interior of \([a, b]\). Hence, \(f\) is monotone on \([a, b]\). By the lemma, \(f\) and \(g\) have a common fixed point in \([a, b]\).

**References**