ELEMENTARY CLASSES CLOSED UNDER DESCENDING INTERSECTION

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The purpose of this note is to establish the following theorem in the theory of models (announced in [**11**]):

THEOREM 1. *If an elementary class* $\mathfrak{K}$ *is closed under descending intersection then* $\mathfrak{K}$ *is a* $\bigwedge_2^0$ *class*; i.e., $\mathfrak{K}$ *is the class of all models of a set of sentences of the form* $(\forall x_0 \cdots x_{m-1})(\exists y_0 \cdots y_{n-1})M$, *where* M *is quantifier-free.*

At the end, the question of extending Theorem 1 to pseudo-elementary classes will be briefly discussed.

1. **Preliminaries.** A structure $\mathfrak{A} = \langle A, R_\xi \rangle_{\xi<\alpha}$ is formed by a nonempty set $A = |\mathfrak{A}|$ and finitary relations R_ξ among the elements of A, for $\xi < \alpha = \lambda(\mathfrak{A})$. If for each ordinal $\xi < \alpha$, R_ξ has ρ_ξ places, the function ρ on α is called the similarity type of $\mathfrak{A}$. If $\mathfrak{B} = \langle B, S_\xi \rangle_{\xi<\alpha}$, $B \subseteq A$, and each S_ξ is the natural restriction of R_ξ to B, then we say $\mathfrak{B}$ is a substructure of $\mathfrak{A}$ ($\mathfrak{B} \subseteq \mathfrak{A}$) and also $\mathfrak{B} = \mathfrak{A}|B$. If $\alpha' + \gamma = \alpha$ and $\mathfrak{A}' = \langle A, R_\xi \rangle_{\xi<\alpha'}$, then we write: $\mathfrak{A}' = \mathfrak{A}|\alpha'$ and $\mathfrak{A} = \langle \mathfrak{A}', R_{\alpha'+\xi} \rangle_{\xi<\gamma}$. We also may write $(A, R_\xi, S_\xi)_{\xi<\alpha}$ to mean $((A, R_\xi)_{\xi<\alpha}, S_\xi)_{\xi<\alpha}$.

The elementary language L_ρ has symbols $\sim$, $\wedge$, $\forall$, $\approx$, and $P_\xi (\xi < \alpha)$. A class $\mathfrak{K}$ of structures of type ρ is called elementary if, for some set Σ of L_ρ sentences, $\mathfrak{K}$ is the class $\operatorname{Mod} \Sigma$ of all models of Σ. $\mathfrak{K}$ is pseudo-elementary if for some elementary class $\mathfrak{K}'$ of a type ρ' extending ρ, $\mathfrak{K} = \{\mathfrak{A}|\alpha / \mathfrak{A} \in \mathfrak{K}'\}$. We write $\mathfrak{A} \equiv \mathfrak{B}$ or $\mathfrak{A} \prec \mathfrak{B}$, respectively, to mean that $\mathfrak{A}$ is elementarily equivalent to $\mathfrak{B}$ or that $\mathfrak{A}$ is an elementary subsystem of $\mathfrak{B}$, in the sense of [**10**] (with elementary = arithmetical). If $\mathfrak{A}|\lambda(\mathfrak{B}) \succ \mathfrak{B}$, we write $\mathfrak{A} \succ_* \mathfrak{B}$ or $\mathfrak{B}_* \prec \mathfrak{A}$.

If $\mathfrak{C}$ is a nonempty set of similar structures, then the notions $\bigcup\{\mathfrak{A}/\mathfrak{A} \in \mathfrak{C}\}$ and $\bigcap\{\mathfrak{A}/\mathfrak{A} \in \mathfrak{C}\}$ are defined in the obvious way, provided that $\mathfrak{C}$ is upward or downward directed, respectively, by the notion $\subseteq$, and provided in the second case that $\bigcap\{|\mathfrak{A}|/\mathfrak{A} \in \mathfrak{C}\} \neq 0$. Under these conditions, we speak of an ascending union or a descending intersection.

Theorem 1 was conjectured by M. Rabin. It improves a result of A. Robinson [**7**]. Robinson showed that an elementary class $\mathfrak{K}$ closed under arbitrary intersection is closed under ascending union; later, Łoś-Suszko [**4**] and Chang [**2**] established that an elementary class

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is Λ_2^0 if and only if it is closed under ascending union. Robinson's proof applies even if $\mathcal{K}$ is closed under pairwise intersection; however, Chang ([1], [2]) showed that such a class is, in fact, closed under arbitrary intersection.

Closure of $\mathcal{K}$ under arbitrary (pairwise) intersection is understood to mean that if $\mathfrak{B}\in\mathcal{K}$ and $\mathfrak{a}\neq 0$ is a (two-element) family of substructures of $\mathfrak{B}$ belonging to $\mathcal{K}$, then $\mathfrak{B}|\bigcap\{|\mathfrak{A}|/\mathfrak{A}\in\mathfrak{a}\}$, if nonempty, is a member of $\mathcal{K}$. The class of structures having exactly two or three elements is an example (pointed out by Rabin) of an elementary class closed under descending, but not arbitrary, intersection.

For elementary classes closed under arbitrary intersection, Rabin [6] has given a detailed syntactical characterization.

2. **Proof of Theorem 1.** Let $\mathcal{K}=\mathrm{Mod}\,\Sigma$ be an elementary class, of type ρ, closed under descending intersection. By the result of Łoś-Suszko [4] and Chang [2], it will suffice to show that $\mathcal{K}$ is closed under ascending union. Suppose $\mathfrak{A}=\bigcup\{\mathfrak{A}'/\mathfrak{A}'\in\mathfrak{a}\}$, where $\mathfrak{a}$ is an upward directed set of members of $\mathcal{K}$.

We shall construct structures $\mathfrak{B}^0, \mathfrak{B}^1, \cdots, \mathfrak{B}^n, \cdots$ with the following properties:

(1) $\mathfrak{A}\subseteq\mathfrak{B}^0$.

(2) $\mathfrak{B}^n=(\mathfrak{B}_0^n, U_1^n, \cdots, U_n^n)$, where $\mathfrak{B}_0^n$ is similar to $\mathfrak{A}$, and $U_1^n, \cdots, U_n^n$ are subsets of $|\mathfrak{B}_0^n|$ (singularly relations). For $0<i\leqq n$, we put $\mathfrak{B}_i^n=\mathfrak{B}_0^n|U_i^n$.

(3) $\mathfrak{B}^n \succ_* \mathfrak{B}^{n-1}$ if $n\neq 0$.

(4) $\mathfrak{B}_n^n\in\mathcal{K}$.

(5) $U_n^n\subseteq U_{n-1}^n$ if $n\neq 0, 1$.

(6) $|\mathfrak{B}^{n-1}|\cap U_n^n=|\mathfrak{A}|$ if $n\neq 0$.

Since every finite substructure of $\mathfrak{A}$ can be embedded in a member of $\mathcal{K}$, $\mathfrak{A}$ can be embedded in $\mathfrak{B}^0\in\mathcal{K}$, by a well-known theorem of Henkin.

Assuming we have $\mathfrak{B}^0, \cdots, \mathfrak{B}^N$ such that (1)–(6) hold for all $n\leqq N$, we shall construct $\mathfrak{B}^{N+1}$ so that (2)–(6) hold for $n=N+1$. Add to the language L_ρ new singularly relation symbols $\boldsymbol{U}_1, \cdots, \boldsymbol{U}_{N+1}$, as well as individual constants $\boldsymbol{b}_\xi$, for $\xi<\gamma$, where $|\mathfrak{B}^N| = \{b_\xi/\xi<\gamma\}$. Let Σ' consist of the following sentences:

(a) All sentences true in $(\mathfrak{B}^N, b_\xi)_{\xi<\gamma}$;

(b) All sentences $\sigma^{\boldsymbol{U}_{N+1}}$, for $\sigma\in\Sigma$ (obtained from σ by relativizing quantifiers to $\boldsymbol{U}_{N+1}$);

(c) $\forall x(\boldsymbol{U}_{N+1}x\rightarrow \boldsymbol{U}_N x)$;

(d) All sentences $\boldsymbol{U}_{N+1}\mathbf{b}_\xi$, for $b_\xi\in|\mathfrak{A}|$, and $\sim\boldsymbol{U}_{N+1}\boldsymbol{b}_\xi$, for $b_\xi\notin|\mathfrak{A}|$.

Any model of Σ' has an isomorph $(\mathfrak{B}^{N+1}, \mathbf{b}_\xi)_{\xi<\gamma}$; then (a), (b), (c), and (d) insure that (3), (4), (5), and (6), respectively, will hold (for $n=N+1$). By the Compactness Theorem, we only need, therefore, to show that an arbitrary finite subset Σ'' of Σ' has a model. The set X of all $b_\xi \in |\mathfrak{A}|$ such that b_ξ occurs in Σ'' is finite; hence $X \subseteq |\mathfrak{A}'|$, for some $\mathfrak{A}' \in \mathfrak{C}$. Clearly, the structure $(\mathfrak{B}^N, |\mathfrak{A}'|)$ is a model of Σ''.[1]

Now let $\mathfrak{B} = \bigcup \{\mathfrak{B}^n / n \in \omega\} = (\mathfrak{B}_0, U_1, \cdots, U_n, \cdots)$. Then $\mathfrak{B}^n {}_* \prec \mathfrak{B}$ for each n. Hence, we obtain from (5), in view of (3), that $U_1 \supseteq U_2 \supseteq \cdots \supseteq U_n \supseteq \cdots$; and from (4) that each $\mathfrak{B}_0 | U_n \in \mathcal{K}$. Since $|\mathfrak{B}|$ is the union of all $|\mathfrak{B}^n|$, (6) insures that $\mathfrak{A} = \bigcap \{\mathfrak{B}_0 | U_n / 0 < n \in \omega\}$. Thus $\mathfrak{A}$ is a descending intersection of members of $\mathcal{K}$, so $\mathfrak{A} \in \mathcal{K}$, as was to be proved.

Recently, D. M. R. Park (see [5] and Thesis, M.I.T., 1964) has obtained some very interesting results including, in particular, a condition which is sufficient as well as necessary for an elementary class $\mathcal{K}$ to be closed under descending intersection. Park gives a different proof of Theorem 1.

3. **Pseudo-elementary classes.** The proof of Theorem 1 can be modified so as to show that a pseudo-elementary class $\mathcal{K}$ closed under descending intersection is closed under ascending union.[2] On the basis of this fact alone, the author stated in [11] that Theorem 1 extends to pseudo-elementary classes. This was a careless fallacy, in view of the well-known fact that a pseudo-elementary class closed under ascending union need not be elementary. Nevertheless, it turns out to be true that:

THEOREM 2. *A pseudo-elementary class $\mathcal{K}$ closed under descending intersection is elementary* (*and hence* $\bigwedge_2^0$ *by Theorem* 1).

This result may be compared with the theorem of Tarski [9] and Łoś [3] that a pseudo-elementary class closed under taking substructures is elementary (and indeed $\bigwedge_1^0$). Theorem 2 was discovered very recently, after the author saw Park's abstract [5]. Though it was not known to Park, it is very closely related to some of his results.

To establish Theorem 2, it suffices, as is well known, to show that $\mathfrak{A} \prec \mathfrak{B} \in \mathcal{K}$ implies $\mathfrak{A} \in \mathcal{K}$. A proof of this fact can be given along the lines of the proof of Theorem 1, but making use of Robinson's consistency theorem [8]. However, we shall not give the details, since a

[1] The construction just given, for the case $N=0$, of $\mathfrak{B}^1$, is exactly the argument in Robinson [7] and Chang [1] showing that $\mathcal{K}$ is $\bigwedge_2^0$ if $\mathcal{K}$ is closed under pairwise intersection.

[2] The corresponding observation for Robinson's and Chang's results concerning nondescending intersection was made by Chang [2].

result somewhat stronger than Theorem 2 can be inferred almost immediately from some of Park's results. A paper by Park will include such an inference (cf. a forthcoming paper by Park).

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