A CLASS OF ENTIRE FUNCTIONS WITH BOWL-LIKE SURFACES

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1. Littlewood and Offord [1] have shown that for "almost all" entire functions of finite order \( p > 0 \), the surface \( u = \log^+ |f(x + iy)| \) in the three-dimensional \( x, y, u \)-space is like a bowl, i.e. outside small pits near the zeros, \( u \) is of order of magnitude of \( \log M(r, f) \) where

\[
M(r, f) = \sup_{|z|=r} |f(z)|.
\]

However, very few of the special functions of analysis are of this type. In this note we show that a large class of functions of the form

\[
f(z) = \prod_{k=1}^{\infty} (1 - (z/r_k)^{m_k})
\]

exhibit this behavior. The choice of the positive integers \( m_k \) and the positive numbers \( r_k \) can be varied in very wide limits. We shall only assume that

2 \( m_{k+1} \geq m_k \),

3 \( \frac{r_{k+1}}{r_k} > 1 + \frac{\delta}{m_k} \) for some \( \delta > 0 \),

4 \( n = o(N(r_n)) \) as \( n \to \infty \),

where

5 \( N(r) = \sum_{r_k < r} m_k \log(r/r_k) \).

It is not hard to see that these conditions can be satisfied in such a way that \( N(r) \) is of the order of magnitude \( r^\rho \) for any assigned positive number \( \rho \).

2. We show now that for \( (r_n-r_{n-1})^{1/2} \leq |z| = r \leq r_n \),

\[
\log |f(z)| = (1 + o(1))N(r) + \log |1 - (z/r)^{m_n}| \quad (r \to \infty).
\]

Now, for

\[
(r_n-r_{n-1})^{1/2} \leq |z| = r < r_n,
\]

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we have

\[
\left| \frac{r_k}{z} \right|^{m_k} < \left( \frac{r_k}{r_{k+1}} \right)^{m_k} (k < \eta - 1),
\]

(7)

\[
< \left( 1 + \frac{\delta}{m_k} \right)^{-m_k} < (1 + \delta)^{-1}
\]

using (3). Similarly

(8) \quad \left| r_{n-1}/z \right|^{m_{n-1}} \pm \left( \frac{r_{n-1}}{r_n} \right)^{m_{n-1}/2} < (1 + \delta)^{-1/2}.

Also, for \( k > n \), by (2) and (3)

(9) \quad \left| z/r_k \right|^{m_k} < \left( \frac{r_n}{r_{n+1}} \right)^{m_n} \cdot \left( \frac{r_{n+1}}{r_{n+2}} \right)^{m_{n+1}} \cdots \left( \frac{r_{k-1}}{r_k} \right)^{m_{k-1}}

\quad < (1 + \delta)^{n-k}.

By (7) and (8)

(10) \quad \sum_{k=1}^{n-1} \log \left| 1 - \left( r_k/z \right)^{m_k} \right| = O(n);

by (9),

(11) \quad \sum_{k=n+1}^{\infty} \log \left| 1 - \left( z/r_k \right)^{m_k} \right| = O(1).

Since

\[
\log |f(z)| = N(r) + \sum_{k=1}^{n-1} \log \left| 1 - \left( r_k/z \right)^{m_k} \right| + \log \left| 1 - \left( z/r_n \right)^{m_n} \right|
\]

\[
+ \sum_{k=n+1}^{\infty} \log \left| 1 - \left( z/r_k \right)^{m_k} \right|
\]

(6) follows from (10), (11) and (4).

For \( r_n \leq |z| = r \leq (r_n \cdot r_{n+1})^{1/2} \), one has

(12) \quad \log |f(z)| = (1 + o(1))N(r) + \log \left| 1 - \left( z/r_n \right)^{m_n} \right| \quad (r \to \infty).

It follows from (6) and (12) that

(13) \quad \log |f(z)| \sim N(r)

as \( z \to \infty \) in any manner outside small circles round the zeros of \( 1 - (z/r_n)^{m_n} \). It is enough to choose the radii of these circles equal to

\[
(r_n/m_n) \cdot \exp(-\epsilon_n \cdot N(r_n)) = \tau_n \quad \text{(say)}
\]

where \( \epsilon_n \to 0 \) as \( n \to \infty \). For \( |1 - (z/r_n)^{m_n}| \) exceeds \( \frac{1}{2} \tau_n \cdot m_n \cdot (r_n)^{-1} \) outside the circular neighborhoods of the zeros when \( n \) is large enough.
3. By placing more stringent conditions on the moduli of the zeros of \( f(z) \), it is possible to relax the symmetrical distribution of their arguments without destroying (13).

Suppose that in addition to the conditions (2), (3) and (4) the sequences \( r_k \) and \( m_k \) satisfy

\[
\frac{r_{k+1}}{r_k} \geq \left(1 + \frac{1}{k}\right)^\sigma \quad \text{for some } \sigma > 1,
\]

(14)

\[
\limsup \log(m_1 + m_2 + \cdots + m_n)/\log r_n = \rho > \frac{1}{\sigma}.
\]

Let \( z_{k,s} \) be equal to \( r_k \exp(i\theta_{k,s}) \), where

\[
2\pi(s - 1)/m_k \leq \theta_{k,s} < 2\pi s/m_k, \quad (s = 1, 2, \cdots, m_k).
\]

Let

\[
\phi_k(z) = \prod_{s=1}^{m_k} \left(1 - \frac{z}{z_{k,s}}\right).
\]

Let the points \( A_1, A_2, \cdots, A_n \) on \( |\xi| = R \) be the corners of a regular polygon of \( n \) sides. Let \( B_1, B_2, \cdots, B_n \) be points on the arcs \( A_1A_2, A_2A_3, \cdots, A_nA_1 \) respectively. Let \( P \) be any point, with affix \( z \), in the \( \xi \)-plane. As \( |\xi - z| \) has but one maximum and one minimum on \( |\xi| = R \), we have

(15)

\[
\frac{\min(PA_k)}{\max(PB_k)} \leq \frac{PA_1 \cdot PA_2 \cdot PA_3 \cdots PA_n}{PB_1 \cdot PB_2 \cdot PB_3 \cdots PB_n} \leq \frac{\max(PA_k)}{\min(PB_k)}.
\]

Hence, for \( r = |z| \neq r_k \),

\[
\left|\frac{r - r_k}{r + r_k}\right| \leq \left|\frac{\phi_k(z)}{1 - (z/r_k)^{m_k}}\right| \leq \left|\frac{r + r_k}{r - r_k}\right|.
\]

Since

\[
\prod_{k=1}^{\infty} \frac{r - r_k}{r + r_k}
\]

is convergent, by hypotheses (14), it follows that

\[
F(z) = \prod_{k=1}^{\infty} \phi_k(z)
\]

is convergent and for

(16)

\[
(r_{n-1}r_n)^{1/2} \leq r \leq (r_nr_{n+1})^{1/2},
\]

we obtain the estimate
\[ \log | F(z) | - \log | f(z) | = O \left( \left| \log \prod_{k=1}^{n-1} \left( r - r_k \right) \left( r + r_k \right) \right| \right) \]

(17)

\[ + O \left( \left| \log \prod_{k=n+1}^{\infty} \left( r_k - r \right) \left( r_k + r \right) \right| \right) + \log \left| p_n(z) \right| - \log \left| 1 - \left( z/r_n \right)^{m_n} \right| . \]

If \( m = n - 1 \), then, by (14)

\[ \frac{1}{n-1} \prod_{k=1}^{n-2} \frac{r - r_k}{r + r_k} > \prod_{k=1}^{m-1} \frac{r_m - r_k}{r_m + r_k} \]

\[ > \prod_{k=1}^{m-1} \left( 1 - \frac{r_k}{r_m} \right)^2 > \prod_{k=1}^{m-1} \left( 1 - \frac{k}{m} \right)^2 \]

\[ > \prod_{k=1}^{m-1} \left( 1 - \frac{k}{m} \right)^2 = \left( \frac{m!}{m^m} \right)^2 > e^{-2m} > e^{-2n}. \]

Also, by (16) and (14) we have

\[ \frac{r - r_n}{r + r_n} > \frac{1}{4} \left( 1 - \frac{r_{n-1}}{r_n} \right) \]

\[ > \frac{1}{4} \left( 1 - \left( \frac{n-1}{n} \right)^{n} \right) \]

\[ > \frac{1}{4n}. \]

Hence

\[ \log \prod_{k=1}^{n-1} \frac{r - r_k}{r + r_k} = O(n). \]

Similarly

\[ \log \prod_{k=n+1}^{2n} \frac{r_k - r}{r_k + r} = O(n). \]

Finally

\[ O > \log \prod_{k>2n} \frac{r_k - r}{r_k + r} > 2 \sum \log \left( 1 - \frac{r}{r_k} \right) > - A \cdot \sum_{k>2n} \left( \frac{r}{r_k} \right) > - A \cdot \sum_{k>2n} \left( \frac{r_{n+1}}{r_k} \right) \]

\[ > - A \sum_{k>2n} \left( \frac{n + 1}{k} \right)^{\sigma} = O(n). \]
Using these estimates in (17), we see that

\[
\log |F(z)| = \log |f(z)| + \log |p_n(z)| - \log \left| 1 - \left( \frac{z}{r_n} \right)^{\alpha_n} \right| + O(n).
\]

Combined with the estimates (15) and (13) this shows that

\[
\log |F(z)| \sim N(r)
\]

outside small pits around the zeros.

**Reference**


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**APPROXIMATE FUNCTIONAL APPROXIMATIONS AND THE RIEMANN HYPOTHESIS**

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1. **Introduction.** Using the functional equation for the Riemann zeta function

\[
\zeta(s) = \chi(s)\bar{\zeta}(1 - s)
\]

where

\[
1/\chi(s) = (2\pi)^{-s} 2 \cos (\pi s/2) \Gamma(s),
\]

it was shown in Spira [1] that

\[
\zeta(s) \neq 0, \quad 1/2 < \sigma < 1, \quad t \geq 10 \quad \text{implies} \quad |\zeta(1 - s)| > |\zeta(s)|
\]

where \( s = \sigma + it \). Using similar but improved techniques, Schoenfeld and Dixon [2] strengthened the result (3) to assuming only \( \sigma > 1/2, \quad |t| \geq 6.8 \) and \( \zeta(s) \neq 0 \). It easily follows from this inequality that the Riemann hypothesis is equivalent to the inequality \( |\zeta(1 - s)| > |\zeta(s)|, \quad 1/2 < \sigma < 1, \quad t \geq 10 \).

Consider now the formula for \( \zeta(s) \) which gives rise to the approximate functional equation and the Riemann-Siegel formula: