# A CLASS OF ENTIRE FUNCTIONS WITH BOWL-LIKE SURFACES

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1. Littlewood and Offord [1] have shown that for "almost all" entire functions of finite order $\rho > 0$, the surface $u = \log^+ |f(x+iy)|$ in the three-dimensional $x$, $y$, $u$-space is like a bowl, i.e. outside small pits near the zeros, $u$ is of order of magnitude of $\log M(r, f)$ where

$$M(r,f) = \sup_{|z|=r} |f(z)|.$$

However, very few of the special functions of analysis are of this type. In this note we show that a large class of functions of the form

$$f(z) = \prod_{k=1}^{\infty} (1 - (z/r_k)^{m_k}) \tag{1}$$

exhibit this behavior. The choice of the positive integers $m_k$ and the positive numbers $r_k$ can be varied in very wide limits. We shall only assume that

$$m_{k+1} \geqq m_k, \tag{2}$$

$$\frac{r_{k+1}}{r_k} > 1 + \frac{\delta}{m_k} \qquad \text{for some } \delta > 0, \tag{3}$$

$$n = o(N(r_n)) \qquad \text{as } n \to \infty, \tag{4}$$

where

$$N(r) = \sum_{r_k \leqq r} m_k \log(r/r_k). \tag{5}$$

It is not hard to see that these conditions can be satisfied in such a way that $N(r)$ is of the order of magnitude $r^\rho$ for any assigned positive number $\rho$.

2. We show now that for $(r_{n-1}r_n)^{1/2} \leqq |z| = r \leqq r_n$,

$$\log |f(z)| = (1 + o(1))N(r) + \log |1 - (z/r)^{m_n}| \qquad (r \to \infty). \tag{6}$$

Now, for

$$(r_{n-1}r_n)^{1/2} \leqq |z| = r < r_n,$$

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we have

$$\left|\frac{r_k}{z}\right|^{m_k} < (r_k/r_{k+1})^{m_k} \qquad (k < n-1), \tag{7}$$
$$< \left(1 + \frac{\delta}{m_k}\right)^{-m_k} < (1+\delta)^{-1}$$

using (3). Similarly

$$| r_{n-1}/z |^{m_{n-1}} \leqq (r_{n-1}/r_n)^{m_{n-1}/2} < (1+\delta)^{-1/2}. \tag{8}$$

Also, for $k > n$, by (2) and (3)

$$| z/r_k |^{m_k} < (r_n/r_{n+1})^{m_n} \cdot (r_{n+1}/r_{n+2})^{m_{n+1}} \cdots (r_{k-1}/r_k)^{m_{k-1}} \tag{9}$$
$$< (1+\delta)^{n-k}.$$

By (7) and (8)

$$\sum_{k=1}^{n-1} \log | 1 - (r_k/z)^{m_k} | = O(n); \tag{10}$$

by (9),

$$\sum_{k=n+1}^{\infty} \log | 1 - (z/r_k)^{m_k} | = O(1). \tag{11}$$

Since

$$\log |f(z)| = N(r) + \sum_{k=1}^{n-1} \log | 1 - (r_k/z)^{m_k} | + \log | 1 - (z/r_n)^{m_n} |$$
$$+ \sum_{k=n+1}^{\infty} \log | 1 - (z/r_k)^{m_k} |,$$

(6) follows from (10), (11) and (4).

For $r_n \leqq |z| = r \leqq (r_n r_{n+1})^{1/2}$, one has

$$\log | f(z) | = (1 + o(1))N(r) + \log | 1 - (z/r_n)^{m_n} | \qquad (r \to \infty). \tag{12}$$

It follows from (6) and (12) that

$$\log | f(z) | \sim N(r) \tag{13}$$

as $z \to \infty$ in any manner outside small circles round the zeros of $1 - (z/r_n)^{m_n}$. It is enough to choose the radii of these circles equal to

$$(r_n/m_n) \cdot \exp(-\epsilon_n \cdot N(r_n)) = \tau_n \text{ (say)}$$

where $\epsilon_n \to 0$ as $n \to \infty$. For $|1 - (z/r_n)^{m_n}|$ exceeds $\frac{1}{2}\tau_n \cdot m_n \cdot (r_n)^{-1}$ outside the circular neighborhoods of the zeros when $n$ is large enough.

3. By placing more stringent conditions on the moduli of the zeros of $f(z)$, it is possible to relax the symmetrical distribution of their arguments without destroying (13).

Suppose that in addition to the conditions (2), (3) and (4) the sequences $r_k$ and $m_k$ satisfy

$$(14)\qquad \begin{aligned} &r_{k+1}/r_k \geqq \left(1+\frac{1}{k}\right)^{\sigma} \qquad \text{for some } \sigma > 1,\\ &\limsup \log(m_1+m_2+\cdots+m_n)/\log r_n = \rho > \frac{1}{\sigma}\cdot \end{aligned}$$

Let $z_{k,s}$ be equal to $r_k \exp(i\theta_{k,s})$, where

$$2\pi(s-1)/m_k \leqq \theta_{k,s} < 2\pi s/m_k, \qquad (s = 1, 2, \cdots, m_k).$$

Let

$$p_k(z) = \prod_{s=1}^{m_k}\left(1 - \frac{z}{z_{k,s}}\right).$$

Let the points $A_1, A_2, \cdots, A_n$ on $|\zeta| = R$ be the corners of a regular polygon of $n$ sides. Let $B_1, B_2, \cdots, B_n$ be points on the arcs $A_1A_2$, $A_2A_3, \cdots, A_nA_1$ respectively. Let $P$ be any point, with affix $z$, in the $\zeta$-plane. As $|\zeta - z|$ has but one maximum and one minimum on $|\zeta| = R$, we have

$$(15)\qquad \frac{\min(PA_k)}{\max(PB_k)} \leqq \frac{PA_1\cdot PA_2\cdot PA_3\cdots PA_n}{PB_1\cdot PB_2\cdot PB_3\cdots PB_n} \leqq \frac{\max(PA_k)}{\min(PB_k)}\cdot$$

Hence, for $r = |z| \neq r_k$,

$$\left|\frac{r-r_k}{r+r_k}\right| \leqq \frac{|p_k(z)|}{|1-(z/r_k)^{m_k}|} \leqq \left|\frac{r+r_k}{r-r_k}\right|.$$

Since

$$\prod_{k=1}^{\infty} \frac{r-r_k}{r+r_k}$$

is convergent, by hypotheses (14), it follows that

$$F(z) = \prod_{k=1}^{\infty} p_k(z)$$

is convergent and for

$$(16)\qquad (r_{n-1}r_n)^{1/2} \leqq r \leqq (r_n r_{n+1})^{1/2},$$

we obtain the estimate

$$
\begin{aligned}
\log |F(z)| - \log|f(z)| &= O\left(\left|\log \prod_{k=1}^{n-1}[(r - r_k)/(r + r_k)]\right|\right) \\
&\quad + O\left(\left|\log \prod_{k=n+1}^{\infty} [(r_k - r)/(r_k + r)]\right|\right) \qquad (17)\\
&\quad + \log| p_n(z)| - \log| 1 - (z/r_n)^{m_n}|.
\end{aligned}
$$

If $m=n-1$, then, by (14)

$$
\begin{aligned}
1 &> \prod_{k=1}^{n-2} \frac{r - r_k}{r+r_k} > \prod_{k=1}^{m-1} \frac{r_m - r_k}{r_m + r_k} \\
&> \prod (1 - (r_k/r_m)^2) > \prod (1 - (k/m)^\sigma)^2 \\
&> \prod_{k=1}^{m-1} (1 - (k/m))^2 = \left(\frac{m!}{m^m}\right)^2 > e^{-2m} > e^{-2n}.
\end{aligned}
$$

Also, by (16) and (14) we have

$$
\begin{aligned}
\frac{r - r_n}{r + r_n} &> \frac{1}{4} (1 - (r_{n-1}/r_n)) \\
&> \frac{1}{4}\left(1 - \left(\frac{n-1}{n}\right)^\sigma\right) \\
&> \frac{1}{4n}\cdot
\end{aligned}
$$

Hence

$$
\log \prod_{k=1}^{n-1} \frac{r - r_k}{r + r_k} = O(n).
$$

Similarly

$$
\log \prod_{k=n+1}^{2n} \frac{r_k - r}{r_k + r} = O(n).
$$

Finally

$$
\begin{aligned}
0 &> \log \prod_{k>2n} \frac{r_k - r}{r_k + r} > 2 \sum \log\left(1 - \frac{r}{r_k}\right) \\
&> - A\cdot \sum_{k>2n} (r/r_k) > - A\cdot \sum_{k>2n} (r_{n+1}/r_k) \\
&> - A \sum_{k>2n} \left(\frac{n+1}{k}\right)^\sigma = O(n).
\end{aligned}
$$

Using these estimates in (17), we see that

$$\log|F(z)| = \log|f(z)| + \log|p_n(z)| - \log\left|1 - \left(\frac{z}{r_n}\right)^{m_n}\right| + O(n).$$

Combined with the estimates (15) and (13) this shows that

$$\log|F(z)| \sim N(r)$$

outside small pits around the zeros.

## REFERENCE

1. J. E. Littlewood and A. C. Offord, *On the distribution of zeros and a-values of a random integral function*. II, Ann. of Math. (2) **49** (1948), 885–952.

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# APPROXIMATE FUNCTIONAL APPROXIMATIONS AND THE RIEMANN HYPOTHESIS

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1. **Introduction.** Using the functional equation for the Riemann zeta function

$$\zeta(s) = \chi(s)\zeta(1-s) \tag{1}$$

where

$$1/\chi(s) = (2\pi)^{-s}\, 2\cos(\pi s/2)\Gamma(s), \tag{2}$$

it was shown in Spira [1] that

$$\zeta(s) \neq 0,\quad 1/2 < \sigma < 1,\quad t \geqq 10 \quad \text{implies} \quad |\zeta(1-s)| > |\zeta(s)| \tag{3}$$

where $s = \sigma + it$. Using similar but improved techniques, Schoenfeld and Dixon [2] strengthened the result (3) to assuming only $\sigma > 1/2$, $|t| \geqq 6.8$ and $\zeta(s) \neq 0$. It easily follows from this inequality that the Riemann hypothesis is equivalent to the inequality $|\zeta(1-s)| > |\zeta(s)|$, $1/2 < \sigma < 1$, $t \geqq 10$.

Consider now the formula for $\zeta(s)$ which gives rise to the approximate functional equation and the Riemann-Siegel formula:

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