A NECESSARY AND SUFFICIENT CONDITION FOR MEMBERSHIP IN $[uv]$

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Levi has obtained [1] for $[y^p]$ and $[uv]$ sufficiency conditions for membership of a power product in the ideal, which tests membership, in certain cases, by a calculation using only the weight and degree of the pp. In this paper we show Levi's conditions for $[uv]$ are necessary as well as sufficient, in contradistinction to $[y^p]$ (see [2]). (This, of course, will show that the answer to Ritt's question [5, p. 177] "What is the least power of u_iv_j which is in $[uv]$?" is $i+j+1$.)

Levi's sufficiency condition can be stated in the following manner: If the pp. P has a negative number in its weight sequence, $P\in[uv]$. Since it is known [2, Theorem III, p. 426] that one need only consider pp. with zero excess weight, for the necessity it will suffice to prove the

THEOREM. *If the pp. P has zero excess weight and a non-negative weight sequence,* $P\notin[uv]$.

We recall some of the definitions of [1] and [2] as well as introduce some notation for this paper.

Let $U(i, r, k)$ represent the product $u_{i_1+r}u_{i_2+r}\cdots u_{i_k+r}$. The signature of $P=U(i, 0, m)V(j, 0, n)$ is (m, n) and the weight of P is $\sum i_\alpha+\sum j_\beta$. For all possible pairs (m', n') where $1\leqq m'\leqq m$, $1\leqq n'\leqq n$, we consider the weight of a factor of P of least weight and signature (m', n'), minus $m'n'$. This set of numbers we call the weight sequence of P. If all the numbers of the weight sequence are non-negative, we say that P has a non-negative weight sequence. The weight of P minus mn is called the excess weight of P.

We facilitate our work by introducing the new variables $\bar{u}_i=u_i/i!$ and $\bar{v}_j=v_j/j!$. For these variables, we have $(\bar{u}_i)'=(i+1)\bar{u}_{i+1}$ and $(\bar{v}_j)'=(j+1)\bar{v}_{j+1}$. To simplify the notation, we write u_i, v_j for $\bar{u}_i$, $\bar{v}_j$ respectively.

If P is a pp. of signature (m, n) and of zero excess weight, then[1] $P\equiv cu^m v_m^n$, and, calling c the multiplier of P, we write $m(P)=c$. Finally, let $D^k=\partial^k/(\partial v_{i_1}\partial v_{i_2}\cdots\partial v_{i_k})$.

LEMMA I. *Assume the pp. A is of zero excess weight and is free of v_0. If $A=u_kP(u, v_1)$, where $P(u, v)$ is a pp. in u and v, then*

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[1] Use [1, Theorem 1.1, p. 543] with the fact that there is a unique α term of the same signature and weight as P.

$$k!m(u_kP(u, v_1)) = (-1)^k \sum m(V(i, 1, k)D^kP(u, v))$$

where the summation is over all ordered sets, $(i_1, \cdots, i_k)$, *of non-negative integers.*

Proof. First we note the equation is meaningful since every term does have zero excess weight. It has been pointed out, [4], that the proof given in [2, Lemma I, p. 429] can be used to show that if $U(i, 0, m)V(j, 0, n)$ has zero excess weight, $m(U(i, 0, m)V(j, 0, n)) = m(u_0U(i, 0, m)V(j, 1, n))$.[2] This constitutes the statement of our lemma for $k=0$. We proceed with the proof using induction on k. Let $A = u_{k+1}P(u, v_1)$ have zero excess weight. Both $u_kP(u, v_1)$ and $V(i, 1, k)D^kP(u, v)$ are in $[uv]$. Thus[3]

$$k!u_kP(u, v_1) \equiv (-1)^k \sum V(i, 1, k)D^kP(u, v)$$

and, taking derivatives of both sides

$$\begin{aligned}(k+1)!u_{k+1}P(u, v_1) &+ k!u_k \sum u_j' \partial P(u, v_1)/\partial u_j + k!u_k \sum v_j' \partial P(u,v_1)/\partial v_j \\ &\equiv (-1)^k \sum V(i, 1, k)u_j' \partial D^kP(u, v)/\partial u_j \\ &+ (-1)^k \sum V(i, 1, k)v_j' \partial D^kP(u, v)/\partial v_j \\ &+ (-1)^k \sum (V(i, 1, k))' D^kP(u, v).\end{aligned}$$

By the induction hypothesis, the multipliers[4] of the first sums on the two sides of the congruence are equal and the multiplier of the second sum of the left equals

$$\begin{aligned}(-1)^k m \sum kV(i, 1, k-1)&v_{i_k+1}(i_k+1)D^{k-1}\partial P(u, v)/\partial v_{i_k-1} \\ &+ (-1)^k m \sum V(i, 1, k)(j+1)v_jD^k\partial P(u, v)/\partial v_{j-1} \\ &= (-1)^k m \sum kV(i, 1, k-1)v_{i_k+2}(i_k+2)D^kP(u, v) \\ &+ (-1)^k m \sum V(i, 1, k)(i_{k+1}+2)v_{i_{k+1}+1}D^{k+1}P(u, v).\end{aligned}$$

Finally, since the second sum on the right equals

$$(-1)^k \sum (i_{k+1}+1)V(i, 1, k+1)D^{k+1}P(u, v)$$

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[2] The referee supplied the following alternative proof. Let h be the isomorphism of $F[u_0, u_1, \cdots; v_0, v_1, \cdots]$ obtained by raising each subscript on a v by 1 and let $h[uv]$ stand for the image of $[uv] = (uv, u_1v+uv_1, u_2v+u_1v_1+uv_2, \cdots)$ under this isomorphism. Then, obviously, $uh[uv] \subset [uv]$. Let $UV \equiv cu^m v_m^n [uv]$. Then $Uh(V) \equiv cu^m v_{m+1}^n (h[uv])$, whence $uUh(V) \equiv cu^{m+1} v_{m+1}^n [uv]$, Q.E.D.

[3] Throughout the proof of Lemma I, each summation is over all ordered sets of nonnegative integers of the symbols $i\alpha$ and $j\beta$ which appear in the terms being summed.

[4] The multiplier of a sum of terms is defined to be the sum of the multipliers of the individual terms.

and the third sum on the right equals

$$(-1)^k \sum kV(i, 1, k-1)(i_k+2)v_{i_{k+2}}D^k(Pu, v),$$

we have

$$(k+1)!m(u_{k+1}P(u, v_1)) + (-1)^k m \sum (i_{k+1}+2)V(i, 1, k+1)D^{k+1}P(w, v)$$
$$= (-1)^k m \sum (i_{k+1}+1)V(i, 1, k+1)D^{k+1}P(u, v)$$

which completes the proof.

LEMMA II. *If the pp. A, of signature (d_1, d_2), has a non-negative weight sequence and zero excess weight, $A \equiv (-1)^t c u^{d_1} v_{d_1}^{d_2}$, where t is the u-weight of A and $c>0$.*

PROOF. If $d_1=1$, A is either $uv_1^{d_2}$ or $u_1 v v_1^{d_2-1}$, and in each case the lemma is easily seen to be true. We complete the proof employing induction on d_1.

We assume for the moment that A is free of v_0 and use Lemma I, noting that every term on the right side of the congruence has u-degree d_1-1 and u-weight k less than the u-weight of A. Thus there is no cancellation among the numbers on the right, as either the induction hypothesis applies or the pp. has a negative term in its weight sequence and, being in $[uv]$, its multiplier is zero.

The proof of this case (A free of v_0) will be complete once we produce a pp. on the right side of the congruence with a non-negative weight sequence. If the v-factor of A is $v_{a_1+1}v_{a_2+1} \cdots v_{a_{d_2}+1}$, $(a_1 \leqq a_2 \leqq \cdots \leqq a_{d_2})$ such a term is

$$Q = V(a, 1, k)\partial^k P(u, v)/(\partial v_{a_1}\partial v_{a_2} \cdots \partial v_{a_k}).$$

Assume this false and let S be a factor of Q with negative excess weight. If we can show that S involves a v_j with $j \geqq a_k+1$, we may assume, without loss of generality, that S is a multiple of $V(a, 1, k)$. Since $V(a, 1, k)$ is a factor of A, we see that S must involve some v_j from the kth partial derivative. Assume $S=UV$ has u-degree$=b$, u-weight$=w_u$ and involves only v_j with $j \leqq a_k$. Then $a_{k+1}=a_k$ and we define r, s, and e by $a_r < a_{r+1} = a_k = a_{k+s} = e < a_{k+s+1}$. Since S is of negative excess weight, $b>e$, and we see that $T=UV(a, 1, r)(v_e)^s$ also has negative excess weight; i.e. $b(s+r) > w_u + a_1 + \cdots + a_r + r + se$. Then $T^* = Uu_k V(a, 1, k+s)$ has excess weight $w_u + k + a_1 + \cdots + a_r + e(s+k-r) + k + s - (b+1)(k+s) < (k-r)(e+1-b) \leqq 0$. But this is a contradiction since T^*, as a factor of A, cannot have negative excess weight.

Thus S must involve a v_j with $j \geqq a_k+1$ and we may assume that S, of signature (m, n) and weight w, is equal to $V(a, 1, k)T(u, v)$. Now

$S^* = V(a, 1, k)T(u, v_1)u_k$ is of signature $(m+1, n)$ and weight $w+n-k+k=w+n$, and since S^* is a factor of A, the weight of $S^* \geqq (m+1)n$. That is, $w+n \geqq (m+1)n$ or $w \geqq mn$. This contradicts our assumption that S was of negative excess weight and consequently there is no such factor of Q. Using the symmetry of $[uv]$, this completes the proof of the theorem of this paper.

To obtain the stronger result of Lemma II, if A involves v_0, we interchange the roles of u and v and find $A \equiv (-1)^r c v^{d_2} u^{d_1}$ where $c > 0$ and r is the v-weight of A. By the theorem of [3],

$$A \equiv (-1)^{r+d_1 d_2} c u^{d_1} v_{d_1}^{d_2},$$

and since A is of excess weight zero, (u-weight of A)+(v-weight of A) $=t+r=d_1 d_2$. Thus $(-1)^{r+d_1 d_2} = (-1)^t$.

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