EXTENSIONS OF COMPLETELY 0-SIMPLE SEMIGROUPS
BY COMPLETELY 0-SIMPLE SEMIGROUPS

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Let $S$ and $T$ be disjoint semigroups, $T$ having a zero element, $O'$. A semigroup $(V, O)$ is called an extension of $S$ by $T$ if it contains $S$ as an ideal and if the Rees factor semigroup $V/S$ is isomorphic to $T$. We shall say that $V$ is determined by a partial homomorphism if there exists a partial homomorphism $A \to \overline{A}$ of $T \setminus O'$ into $S$ such that $A \circ B = AB$ if $AB \neq O'$, $A \circ B = \overline{AB}$ if $AB = O'$, $A \circ s = \overline{As}$, $s \circ A = s\overline{A}$, and $s \circ t = st$ where $s, t$ in $S$ and the operations in $S$ and $T$ are denoted by juxtaposition. The purpose of this note is to give a necessary and sufficient condition that $V$ be determined by a partial homomorphism when $S$ is a completely 0-simple semigroup and $T$ is a completely 0-simple semigroup. Since these partial homomorphisms are known mod group homomorphisms [1, p. 109, Theorem 3.14], our extensions may be given an explicit form. A corollary to our theorem will include an important theorem due to W. D. Munn [1, p. 143, Theorem 4.22]. Our result should have important applications to the study of finite semigroups and to semigroups with some finiteness condition.

If $S$ is any subset of a semigroup, $\mathcal{E}(S)$ will denote the set of idempotents of $S$ and $S^*$ will denote the set of nonzero elements of $S$. $\mathfrak{R}$, $\mathfrak{L}$, $\mathfrak{D}$ and $\mathfrak{C}$ will denote Green's relations [1, p. 47]. If $a \in S$, $R_a$ will denote the $\mathfrak{R}$-class containing $a$. If $e$ and $f$ are idempotents and $ef = fe = e$, we say $e$ is under $f$ and write $e < f$. Basic definitions are given in [1]. Likewise references to the fundamental work of Clifford, Green, and Munn will be found in [1].

**Lemma.** Let $V$ be an extension of a completely 0-simple semigroup $S$ by a completely 0-simple semigroup $T$, and let $0$ be the zero of $S$ (hence, $0$ is also the zero of $V$). If there is some $E \in \mathcal{E}(T^*)$ such that $ESE = 0$, then $V$ is given by the partial homomorphism $\xi: T^* \to S$ which maps every element of $T^*$ to $0$.

**Proof.** Let $E \in \mathcal{E}(T^*)$ such that $ESE = 0$. If $F \in \mathcal{E}(T^*)$, there exists $Y \in T^*$ such that $E \mathfrak{R} Y$ and $Y \mathfrak{L} F$ [1, p. 79, Theorem 2.51]. Thus,
$EY = Y$ and there exists $U \in T^*$ such that $UY = F$. Let $s \in S$. Now, $ESY = ESEY = 0$, $EYS = 0$, $YSY = 0$, $UYS = 0$, $FSY = 0$, $FSF = 0$, and $FSF = 0$. Thus, $FSF = 0$ for all $F \in E(T^*)$. Now suppose that $A \in R_E \cap L_F$ where $E, F \in E(T^*)$. Let $h \in S$ and $hA \in Lg$ for some $g \in E_S$. Since $uhA = g$ for some $u \in S$, $gF = uhAF = g$. Thus, $FG = FgF = 0$. Hence $gFg = g = g = g$ and $hA = 0$. Similarly $Ah = 0$.

Now define $A\xi = 0$ for all $A \in T^*$. Clearly $\xi$ is a partial homomorphism of $T^*$ into $S$. If $AB = s \in S$, $AB = (EA)B = Es = 0 = (A\xi)(B\xi)$.

**Theorem.** An extension $V$ of a completely 0-simple semigroup $S$ by a completely 0-simple semigroup $T$ is given by a partial homomorphism if and only if under each nonzero idempotent of $T$ there exists at most one nonzero idempotent of $S$.

**Proof.** Suppose that under each nonzero idempotent of $T$ there exists at most one nonzero idempotent of $S$. By virtue of the lemma, we may assume that $ESE \neq 0$ for all $E \in E(T^*)$. We will first show that under each nonzero idempotent $E \in T^*$, there exists a unique idempotent $e \in S^*$ and that $ESE = H_eU_0$. Let $a \in ESE$ and $a \neq 0$. There exists $x \in ESE$ such that $axa = a$. Thus $ax = e \in E(S^*)$ and $e < E$, i.e., $e$ is the unique idempotent of $S^*$ which lies under $E$. Hence $xa = e$ and $ESE \subseteq H_eU_0$. If $b \in H_e, bE = beE = be = b = Eb$ and $ESE = H_eU_0$. Let $A \in R_B \cap L_F$ where $E, F \in E(T^*)$, and let $e$ and $f$ be the unique idempotents of $S^*$ under $E$ and $F$ respectively. We will show that if $h \in S, hA \neq 0$ implies that $hA \in L_f$ and $Ah \neq 0$ implies that $Ah \in R_e$. We consider only the first case, the other case being similar. Now, $hA \in Lg$ for some $g \in E(S^*)$. As in the proof of the lemma, we show that $gF = g$. There exists $k \in S^*$ such that $gF = kg = k = fF = Ff = Fk$ and $k \in FSF$. Hence $k \in H_f$ and $hA \in L_f$. Next suppose that $A$ is also an element of $R_B \cap L_F$, where $E'$ and $F' \in E(T^*)$ and let $e'$ and $f'$ denote the unique idempotents under $E'$ and $F'$ respectively. We will show that $eAf = e' Af'$ and hence it will follow that we may write $A\xi = eAf$ where $\xi$ is a single valued mapping of $T^*$ into $S$. We first note that $FF' = F, F'F = F', FF' = f$, and $FF' = f'$. Thus, $FSFF'F' = FSFSF'$. Since $f \in SFS, SFS = S$, and $f = FfF' \in FSF F' SF' = H_f H_f \cup 0$. Thus $H_f H_f \neq 0$ and $ff' \in H_f H_f = R_f \cap L_f \subseteq S^*$ [1, p. 79, Theorem 2.52]. Since $ff' = Fff'F \in FSF, ff' \in H_f, ff'f = ff', (ff')^2 = ff'$, and $ff' = f$. Similarly, $ff' = f'$, i.e. $f \neq f'$. In an analogous manner, we show that $e \neq e'$. If $e'EE = 0, e'EE = 0$ and $e' = 0$. Hence $e'EE = 0$ and similarly $Ff' \neq 0$. Thus, $eAf = eAf = eAf = eAf$. We next show that $\xi$ is a partial homomorphism of $T^*$ into $S$. Let $B \in R_G \cap L_H$ where $G, H \in E(T^*)$, and let $g$ and $h$ denote the unique idempotents of $S^*$ under
G and H respectively. If \( AB \neq O' \), \( AB \in R_E \cap L_H \) [1, p. 79, Theorem 2.52]. Thus \((AB)\xi = eA Bh\) and \( A\xi B\xi = eAfg Bh = [(eA)f][g(Bh)]\). Since \( A \in E \), \( eA \in e \) and \( eA \neq 0 \). Similarly, \( B\xi \neq 0 \) and hence \((AB)\xi = A\xi B\xi\). Now, let \( b \in S \). If \( eAb \neq 0 \), \( eAb = ((eA)f)b = (eA)b = e(AB) = Ab \). If \( Ab \neq 0 \), we may reverse the steps. Thus \( Ab = (A\xi)b \) in all cases. Similarly \( bA = b(A\xi) \). Then, \( s = sH = sh = (AB)h \). If \( s \neq 0 \), \( A(Bh) = A\xi Bh = A\xi g Bh = A\xi B\xi \) and \( AB = A\xi B\xi \neq 0 \). If \( A\xi B\xi \neq 0 \), \( A\xi B\xi = ((eA)f)(g(Bh)) = eA(g(Bh)) = ((eA)g)(Bh) = (e(Ag))(Bh) = (Ag)(Bh) = A(g(Bh)) = A(Bh) = s \). Thus, in all cases, \( AB = A\xi B\xi \).

Conversely, suppose that the extension \( V \) is given by a partial homomorphism \( \xi \) of \( T^* \) into \( S \). First suppose that \( A\xi = 0 \) for some \( A \in T^* \). Let \( U = A \in T^*: A\xi = 0 \) and let \( U' = U \cup 0 \). Clearly, \( U' \) is a nonzero ideal of \( T \) and hence \( U' = T \), i.e., \( A\xi = 0 \) for all \( A \in T^* \). In this case, if \( E \in \varepsilon(T^*) \), \( 0 \) is the only idempotent under \( E \). If \( A\xi \neq 0 \) for all \( A \in T^* \), \( E\xi \) is the unique idempotent of \( S^* \) under \( E \).

**Corollary.** An extension \( V \) of a completely simple semigroup \( S \) by a completely 0-simple semigroup \( T \) is given by a partial homomorphism if and only if under each nonzero idempotent of \( T \) there exists at most one idempotent of \( S \).

**Remark.** In the statement of the theorem, we may replace \( T \) by a regular semigroup with zero in which every nonzero idempotent is primitive [1]. This follows since such a semigroup is an orthogonal sum of completely 0-simple semigroups [2].

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**References**


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