EXTENSIONS OF COMPLETELY 0-SIMPLE SEMIGROUPS
BY COMPLETELY 0-SIMPLE SEMIGROUPS

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Let $S$ and $T$ be disjoint semigroups, $T$ having a zero element, $O'$. A semigroup $(V, O)$ is called an extension of $S$ by $T$ if it contains $S$ as an ideal and if the Rees factor semigroup $V/S$ is isomorphic to $T$. We shall say that $V$ is determined by a partial homomorphism if there exists a partial homomorphism $A \to A$ of $T \setminus O'$ into $S$ such that $A \circ B = AB$ if $AB \neq O'$, $A \circ B = AB$ if $AB = O'$, $A \circ s = As$, $s \circ A = sA$, and $s \circ t = st$ where $s, t$ in $S$ and the operations in $S$ and $T$ are denoted by juxtaposition. The purpose of this note is to give a necessary and sufficient condition that $V$ be determined by a partial homomorphism when $S$ is a completely 0-simple semigroup and $T$ is a completely 0-simple semigroup. Since these partial homomorphisms are known mod group homomorphisms [1, p. 109, Theorem 3.14], our extensions may be given an explicit form. A corollary to our theorem will include an important theorem due to W. D. Munn [1, p. 143, Theorem 4.22]. Our result should have important applications to the study of finite semigroups and to semigroups with some finiteness condition.

If $S$ is any subset of a semigroup, $\mathcal{E}(S)$ will denote the set of idempotents of $S$ and $S^*$ will denote the set of nonzero elements of $S$. $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{H}$ will denote Green's relations [1, p. 47]. If $a \in S$, $R_a$ will denote the $\mathcal{R}$-class containing $a$. If $e$ and $f$ are idempotents and $ef = fe = e$, we say $e$ is under $f$ and write $e < f$. Basic definitions are given in [1]. Likewise references to the fundamental work of Clifford, Green, and Munn will be found in [1].

**Lemma.** Let $V$ be an extension of a completely 0-simple semigroup $S$ by a completely 0-simple semigroup $T$, and let $0$ be the zero of $S$ (hence, 0 is also the zero of $V$). If there is some $E \in \mathcal{E}(T^*)$ such that $ESE = 0$, then $V$ is given by the partial homomorphism $\xi: T^* \to S$ which maps every element of $T^*$ to 0.

**Proof.** Let $E \in \mathcal{E}(T^*)$ such that $ESE = 0$. If $F \in \mathcal{E}(T^*)$, there exists $Y \in T^*$ such that $E \mathcal{R} Y$ and $Y \mathcal{L} F$ [1, p. 79, Theorem 2.51]. Thus,
$EY = Y$ and there exists $U \in T^*$ such that $UY = F$. Let $s \in S$. Now, $ESY = ESEY = 0$, $EYSY = 0$, $UYSY = 0$, $FSY = 0$, $FSF = 0$. Thus, $FSF = 0$ for all $F \in \mathcal{E}(T^*)$. Now suppose that $A \in R_F \cap L_F$ where $E, F \in \mathcal{E}(T^*)$. Let $h \in S$ and $hA \in L_F$ for some $g \in E_S$. Since $uhA = g$ for some $u \in S$, $gF = uhAF = g$. Thus, $Fg = FgF = 0$. Hence $gFg = 0 = gg = g$ and $hA = 0$. Similarly $Ah = 0$. Now define $A\xi = 0$ for all $A \in T^*$. Clearly $\xi$ is a partial homomorphism of $T^*$ into $S$. If $AB = s \in S$, $AB = (EA)B = Es = 0 = (A\xi)(B\xi)$.

**Theorem.** An extension $V$ of a completely $0$-simple semigroup $S$ by a completely $0$-simple semigroup $T$ is given by a partial homomorphism if and only if under each nonzero idempotent of $T$ there exists at most one nonzero idempotent of $S$.

**Proof.** Suppose that under each nonzero idempotent of $T$ there exists at most one nonzero idempotent of $S$. By virtue of the lemma, we may assume that $ESE \neq 0$ for all $E \in \mathcal{E}(T^*)$. We will first show that under each nonzero idempotent $E \in T^*$, there exists a unique idempotent $e \in S^*$ and that $ESE = H_eU_0$. Let $a \in ESE$ and $a \neq 0$. There exists $x \in ESE$ such that $axa = a$. Thus $ax = e \in \mathcal{E}(S^*)$ and $e < E$, i.e., $e$ is the unique idempotent of $S^*$ which lies under $E$. Hence $ax = e$ and $ESE \subseteq H_eU_0$. If $b \in H_e$, $bE = beE = be = b = Eb$ and $ESE = H_eU_0$. Let $A \in R_E \cap L_E$ where $E, F \in \mathcal{E}(T^*)$, and let $e$ and $f$ be the unique idempotents of $S^*$ under $E$ and $F$ respectively. We will show that if $h \in S$, $hA \neq 0$ implies that $hA \subseteq L_f$ and $Ah \neq 0$ implies that $Ah \subseteq L_e$. We consider only the first case, the other case being similar. Now, $hA \subseteq L_g$ for some $g \in \mathcal{E}(S^*)$. As in the proof of the lemma, we show that $gF = g$. There exists $k \in S^*$ such that $g \not\subseteq k$ and $k \not\subseteq f$. Thus, $kF = kgF = kg = kfk = Ffk = Fk$ and $k \subseteq FSF$. Hence $k \subseteq H_f$ and $hA \subseteq L_f$. Next suppose that $A$ is also an element of $R_E \cap L_E$, where $E'$ and $F' \in \mathcal{E}(T^*)$ and let $e'$ and $f'$ denote the unique idempotents under $E'$ and $F'$ respectively. We will show that $eAf = e'Af'$ and hence it will follow that we may write $A\xi = eAf$ where $\xi$ is a single valued mapping of $T^*$ into $S$. We first note that $FF' = F$, $F'F = F'$, $fF' = f$, and $f'f = f'$. Thus, $FSF'\subseteq FSFSF'$. Since $f \in SFS$, $SFS = S$, and $f = ff' \in FSF' \subseteq FSFSF' = H_fH_f \cup 0$. Thus $H_fH_f \neq 0$ and $ff' \in H_fH_f = R_f \cap L_f \subseteq S^*$ [1, p. 79, Theorem 2.52]. Since $ff' = Fff' \subseteq FSF$, $ff' \subseteq H_f$, $ff'f = ff'$, $(ff')^2 = ff'$, and $ff' = f$. Similarly, $f'f = f'$, i.e. $f \not\subseteq f'$. In an analogous manner, we show that $e \not\subseteq e'$. If $e'F = 0$, $e'EE' = 0$ and $e' = 0$. Hence $e'F \neq 0$ and similarly $Ff' \neq 0$. Thus, $e'Af' = e'e'EAFF'f = e(e'E)A(Ff')f = e(e'E)eAf(Ff')f = eAf$. We next show that $\xi$ is a partial homomorphism of $T^*$ into $S$. Let $B \in R_g \cap L_H$ where $G, H \in \mathcal{E}(T^*)$, and let $g$ and $h$ denote the unique idempotents of $S^*$ under
G and H respectively. If $AB \neq O'$, $AB \in R_E \cap L_H$ [1, p. 79, Theorem 2.52]. Thus $(AB) \xi = eABh$ and $A\xi B\xi = eAfgBh = [(eA)f][g(Bh)]$. Since $A \in E$, $eA \in e$ and $eA \neq 0$. Similarly, $Bh \neq 0$ and hence $(AB) \xi = A\xi B\xi$. Now, let $b \in S$. If $eAfb \neq 0$, $eAfb = ((eA)f)b = (eA)b = e(AB) = Ab$. If $Ab \neq 0$, we may reverse the steps. Thus $Ab = (A\xi)b$ in all cases. Similarly $bA = b(A\xi)$. Now suppose that $AB = s \in S$. Then, $s = sH = sh = (AB)h$. If $s \neq 0$, $A(Bh) = A\xi Bh = A\xi gBh = A\xi B\xi$ and $AB = A\xi B\xi \neq 0$. If $A\xi B\xi \neq 0$, $A\xi B\xi = ((eA)f)(g(Bh)) = eA(g(Bh)) = ((eA)g)(Bh) = (e(Ag))(Bh) = (Ag)(Bh) = A(g(Bh)) = A(Bh) = s$. Thus, in all cases, $AB = A\xi B\xi$.

Conversely, suppose that the extension $V$ is given by a partial homomorphism $\xi$ of $T^*$ into $S$. First suppose that $A\xi = 0$ for some $A \in T^*$. Let $U = (A \in T^* : A\xi = 0)$ and let $U' = U \cup 0$. Clearly, $U'$ is a nonzero ideal of $T$ and hence $U' = T$, i.e., $A\xi = 0$ for all $A \in T^*$. In this case, if $E \in \mathcal{E}(T^*)$, 0 is the only idempotent under $E$. If $A\xi \neq 0$ for all $A \in T^*$, $E\xi$ is the unique idempotent of $S^*$ under $E$.

**Corollary.** An extension $V$ of a completely simple semigroup $S$ by a completely 0-simple semigroup $T$ is given by a partial homomorphism if and only if under each nonzero idempotent of $T$ there exists at most one idempotent of $S$.

**Remark.** In the statement of the theorem, we may replace $T$ by a regular semigroup with zero in which every nonzero idempotent is primitive [1]. This follows since such a semigroup is an orthogonal sum of completely 0-simple semigroups [2].

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**References**


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