Using these estimates in (17), we see that
\[ \log |F(z)| = \log |f(z)| + \log |p_n(z)| - \log \left| 1 - \left( \frac{z}{r_n} \right)^m \right| + O(n). \]

Combined with the estimates (15) and (13) this shows that
\[ \log |F(z)| \sim N(r) \]
outside small pits around the zeros.

Reference


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APPROXIMATE FUNCTIONAL APPROXIMATIONS AND THE RIEMANN HYPOTHESIS

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1. Introduction. Using the functional equation for the Riemann zeta function
\[ \xi(s) = \chi(s)\xi(1 - s) \]
where
\[ 1/\chi(s) = (2\pi)^{-s} 2 \cos (\pi s/2) \Gamma(s), \]
it was shown in Spira [1] that
\[ \xi(s) \neq 0, \quad 1/2 < \sigma < 1, \quad t \geq 10 \quad \text{implies} \quad |\xi(1 - s)| > |\xi(s)| \]
where \( s = \sigma + it \). Using similar but improved techniques, Schoenfeld and Dixon [2] strengthened the result (3) to assuming only \( \sigma > 1/2, \quad |t| \geq 6.8 \) and \( \xi(s) \neq 0 \). It easily follows from this inequality that the Riemann hypothesis is equivalent to the inequality \( |\xi(1 - s)| > |\xi(s)|, \quad 1/2 < \sigma < 1, \quad t \geq 10 \).

Consider now the formula for \( \xi(s) \) which gives rise to the approximate functional equation and the Riemann-Siegel formula:
\[
(4) \quad \xi(s) = g_m(s) + \frac{e^{ixs} \Gamma(1 - s)}{2\pi i} \int_c \frac{z^{s-1} e^{-mw}}{e^w - 1} \, dw
\]

where

\[
(5) \quad g_m(s) = \sum_{n=1}^{m} n^{-s} + x(s) \cdot \sum_{n=1}^{m} n^{s-1}.
\]

The \( g_m(s) \) are the approximate functional approximations of the title, and, as noted implicitly by Titchmarsh ([4, p. 74]), they satisfy the same functional equation as \( \xi(s) \). Hence, just as in the case of the \( \xi \)-function, \( g_m(s) \) has its zeros on the critical line for \( |t| > 6.8 \) if and only if \( |g_m(1-s)| > |g_m(s)| \).

It is thus natural to write (4) in the form

\[
\xi(s) = g_m(s) + B
\]

and study the location of the zeros of \( g_m(s) \), (hopefully on the critical line), and attempt to carry the final conclusion of the Riemann hypothesis via the ideas of Rouche.

It is indeed possible to show that \( g_1(s) \) and \( g_2(s) \) have their zeros on the critical line (for \( t \) sufficiently large) and this proof is carried out in §3, with the aid of two lemmas in §2.

Massive calculations were undertaken to verify the hypothesis for \( m \geq 3 \), but these calculations instead revealed a remarkable scientific situation, which reinforces the possibility of using Rouche’s theorem. The evidence strongly suggests the conjecture: If \( m \geq 3 \), and \( s \) is in the critical strip, then \( g_m(s) \) has its zeros on the critical line for \( (2\pi m)^{1/2} \leq t \leq 2\pi m \), and has zeros off the line outside this interval. The computations supporting this conjecture will be reported in full in another paper.

2. Lemmas on \( \chi(s) \). We write \( D \) for \( d/ds \) and \( D_\sigma \) for \( \partial/\partial \sigma \).

**Lemma 1.** If \( |t| \geq 10 \) and \( \sigma > 1/2 \) then \( D_\sigma \log |1/\chi(s)| > \log |s| - 1.93 \).

**Proof.** From Schoenfeld-Dixon [2], we have \( D_\sigma \log |1/\chi(s)| > \log |s| - |s|^{-1/2} - |s|^{-2/12} - |t|^{-3/5} - (\log 2\pi + \pi/(4 \sinh^2(\pi t/2))) \) from which the lemma easily follows.

**Lemma 2.** If \( |t| \geq 10 \) and \( \sigma > 1/2 \) then

\[
|1/\chi(s)| > .9646(|s|/(2\pi))^{\sigma-1/2}.
\]

**Proof.** We have

\[
(6) \quad |1/\chi(s)| = |2\pi|^{-\sigma} |2 \cos(\pi s/2)| |\Gamma(s)|.
\]
As shown in Spira [1],

$$|2 \cos(\pi s/2)| \geq 2 \sinh(\pi t/2) = e^{\pi t/2} - e^{-\pi t/2} > 0.99 e^{\pi t/2},$$

the last inequality holding for $t \geq 10$. Also from Spira [1] we have

$$|\Gamma(s)| = (2\pi)^{1/2} e^{-\sigma} |s|^{\sigma-1/2} e^{-t \arg s} |e^{1/(12s)} + R_1|$$

where $|R_1| < |s|^{-1/6}$. It is easy to see that if $|s| < 1$, then $|e^s| \geq 1 - |s| [1/(1 - |s|)].$ Now $1/(12s) + R_1 < |s|^{-1/12} + |s|^{-1/6} = |s|^{-1/4} \leq 1/40$ if $t \geq 10$. Hence, setting $z = 1/(12s) + R_1$,

$$|e^{1/(12s)} + R_1| \geq 1 - |z| [1/(1 - |z|)] \geq 38/39$$

the last inequality holding since $|z| < 1/40$. By elementary geometry $t(\pi/2 - \arg s) > \sigma$, so the lemma follows on combining equations (6)–(9).

3. The cases $m = 1, 2$.

**Theorem** For $m = 1, 2$ and $|t|$ sufficiently large, $g_m(s)$ has all its complex zeros on $\sigma = 1/2$.

**Proof.** For $m = 1$ we have $g_m(s) = 1 + \chi(s)$, and for $\sigma > 1/2$ and $|t| > 6.8$, by Schoenfeld-Dixon [2], we have $|g_1(s)| \geq 1 - |\chi(s)| > 0$. An easy argument shows that $g_1(s)$ has exactly one zero in each Gram interval.

For $m = 2$, $|g_m(s)| \geq |1 + 2^{-s}| - |\chi(s)| \cdot |1 + 2^{s-1}|$, and $|g_1(s)| > 0$ provided

$$1/\chi(s) > |(1 + 2^{s-1})/(1 + 2^{-s})|.$$

On $\sigma = 1/2$ both sides of (10) are 1, so that proceeding as in Schoenfeld-Dixon [2], (10) will hold provided

$$D_\sigma \log |1/\chi(s)| > D_\sigma \log |(1 + 2^{s-1})/(1 + 2^{-s})|.$$

Since (Schoenfeld-Dixon [2]) $D_\sigma \log |f(s)| = \text{Re } D \log f(s),$

$$D_\sigma \log \left|\frac{1 + 2^{s-1}}{1 + 2^{-s}}\right| = \log 2 \text{Re } \left[\frac{1 + 2^{-s} + 2^{s-1}}{(1 + 2^{-s})(1 + 2^{s-1})}\right]$$

$$\leq \log 2 \left|\frac{1 + 2^{-s} + 2^{s-1}}{(1 + 2^{-s})(1 + 2^{s-1})}\right|$$

$$\leq \log 2 \left[\frac{1 + 2^{-\sigma} + 2^{\sigma-1}}{(1 - 2^{-\sigma})(1 - 2^{\sigma-1})}\right]$$

where we must now take $1/2 < \sigma < 3/4$ to obtain a bound on the denominator. The numerator $1 + 2^{-\sigma} + 2^{\sigma-1}$ has a minimum at $\sigma = 1/2,$
and for $1/2 < \sigma < 1$ rises monotonely from $1 + (2)^{1/2}$ to 2.5. The denominator is $1.5 - (2^{-\sigma} + 2^{\sigma-1})$ which is smallest at $\sigma = 3/4$. Thus $D_\sigma \left| \log(1 + 2^{\sigma-1})/(1 + 2^{-\sigma}) \right| < 2.5 \log 2 \left/ [1.5 - (2^{-3/4} + 2^{-1/4})] \right. < 27$. Using Lemma 1, we need only choose $|s|$ so large that $\log |s| > 1.93 + 27$, i.e., $t > e^{29}$.

For $\sigma > 3/4$, we proceed directly from (10) using Lemma 2. We have $\left| (1 + 2^{\sigma-1})/(1 + 2^{-\sigma}) \right| \leq (1 + 2^{\sigma-1})/(1 - 2^{-3/4})$ so that (10) will hold provided

$$\frac{0.9646(|s|/(2\pi))^{\sigma-1/2}}{1 + 2^{\sigma-1}} > (1 + 2^{\sigma-1})/(1 - 2^{-3/4}).$$

For $3/4 \leq \sigma \leq 1$, the right hand side of (12) is bounded by 5, and an easy calculation shows we need only take $t > 2\pi \cdot 64 \sim 8145$. For $\sigma > 1$, $1 + 2^{\sigma-1} < 2^{\sigma}$, so (12) transforms to $(|s|/(4\pi)^{\sigma-1/2}) > (2)^{1/2}/0.376$, which will be valid if $t > 4\pi((2)^{1/2}/0.376)^4 \sim 805$. This completes the proof of the theorem of this section.

Since there is empirically a steady appearance of zeros off the critical line for $m \geq 3$, it appears unlikely that one would be able to extend the theorem of this section to any further $m$.

**References**


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