A NOTE ON SPLITTING IN SOLVABLE GROUPS

ERNEST E. SHULT

1. Introduction. The theorem presented below generalizes theorems of E. Schenkman [5] and G. Higman [4] concerning splitting in finite solvable groups. This generalization is achieved by applying results from the theory of formations recently developed by W. Gaschütz [2], [3]. All groups considered here are finite and solvable. A formation, $\mathcal{F}$, is a collection of groups closed under taking homomorphisms and subdirect products. It follows that every group, $G$, contains a characteristic subgroup, $G_\mathcal{F}$, minimal with respect to the property that $G/G_\mathcal{F} \in \mathcal{F}$. A formation, $\mathcal{F}$, is called saturated, if $G/\phi(G) \in \mathcal{F}$ implies $G \in \mathcal{F}$ for all $G$ (see [3]). $F$ is called an $\mathcal{F}$-subgroup of $G$ if $F \in \mathcal{F}$ and if $F \leq H \leq G$ implies $FH_\mathcal{F} = H$. A theorem of Gaschütz [2], states that if $\mathcal{F}$ is saturated, $\mathcal{F}$-subgroups of $G$ always exist and are conjugate in $G$.

Theorem. Let $\mathcal{F}$ be a saturated formation and suppose that for a finite solvable group, $G$, $G_\mathcal{F}$ is abelian. Then:

(i) the $\mathcal{F}$-subgroups of $G$ complement $G_\mathcal{F}$.

(ii) all complements of $G_\mathcal{F}$ in $G$ are conjugate and hence are $\mathcal{F}$-subgroups of $G$.

Let $L_0(G) = G$ and let $L_i(G)$ be the $i$th term of the lower nilpotent series of $G$. If $\mathcal{F}$ denotes the formation of groups having nilpotent length $\leq k - 1$, the theorem yields a theorem of Higman [4] which states that if $L_k(G)$ is abelian, $G$ splits over $L_k(G)$ and all complements of $L_k(G)$ in $G$ are conjugate. (This statement becomes a theorem of Schenkman [5] when $k = 2$.) R. Carter [1] was able to identify the complements in Higman's theorem as the relative system normalizers of $L_{k-1}(G)$ in $G$. From (ii) we may also identify them as the $\mathcal{F}$-subgroups of $G$ (or as the Carter subgroups of $G$ when $k = 2$). Our theorem yields a number of other interesting results on splitting when $\mathcal{F}$ ranges over various saturated formations, for example, groups having nilpotent commutator subgroups, supersolvable groups, groups $G$, for which $G/G'$ is a $\pi$-group, etc.

2. Preliminary results and proof of the theorem. Our proof employs a number of basic results of Gaschütz. First if $F$ is an $\mathcal{F}$-subgroup...
of \( G \), \( F \subseteq H \subseteq G \) implies that \( F \) is an \( \Phi \)-subgroup of \( H \). Also if \( N \) is normal in \( G \), \( FN/N \) is an \( \Phi \)-subgroup of \( G/N \) and every \( \Phi \)-subgroup of \( G/N \) has the form \( F_0N/N \) where \( F_0 \) is an \( \Phi \)-subgroup of \( G \). Let \( \pi \) and \( \pi' \) denote a partition of the set of primes. \( O_{\pi'}(G) \) denotes the maximal normal \( \pi' \)-subgroup of \( G \) and \( O_{\pi'\pi}(G) \) denotes the subgroup of \( G \) for which \( O_{\pi'\pi}(G)/O_{\pi'}(G) \) is the maximal normal \( \pi \)-subgroup of \( G/O_{\pi'}(G) \). A formation, \( \Phi \), is said to be locally defined if for some sequence of (possibly empty) formations, \( f(\pi) \), \( \pi \) ranging over the primes, \( G \in \Phi \) if and only if there is \( \pi \) ranging over the primes, \( G \in \Phi \) if and only if \( \pi \) ranging over the primes, \( G \in \Phi \) if and only if \( \pi \).\( \phi \) \subseteq \( G \) otherwise. Gaschütz proved \( [2] \) that all locally defined formations are saturated and recently has announced \( 2 \) the important result that, conversely, all saturated formations are locally defined by some sequence of local formations \( \{ f(\pi) \} \).

**Proof of the theorem.** (i) If \( G_\Phi = E \), \( G \) is its own \( \Phi \)-subgroup and the theorem is trivial. Suppose then that \( G_\Phi \neq E \). Let \( F \) be an \( \Phi \)-subgroup of \( G \). Since \( FG_\Phi = G \), to prove (i) it suffices to show that \( FC \subseteq G_\Phi \neq E \). Suppose \( FC \subseteq G_\Phi \neq E \). Then since \( G_\Phi \) is abelian, \( F \subseteq G_\Phi \) is normal in \( G \). Let \( N \) be a minimal normal subgroup of \( G \). Then \( (G/N)_\Phi = G_\Phi N/N \) is abelian and \( FN/N \) is an \( \Phi \)-subgroup of \( G/N \). By induction, \( FN/N \subseteq G_\Phi N \subseteq N \). Thus \( FC \subseteq G_\Phi \subseteq N \). It follows that \( F \subseteq G_\Phi = N_\Phi \), the unique minimal normal subgroup of \( G \).

Suppose \( N_\Phi = G_\Phi \). Then \( G = FN_\Phi = F \subseteq \Phi \) whence \( G_\Phi = E \), a contradiction. Hence \( N_\Phi \subseteq G_\Phi \).

Suppose \( F \subseteq H \subseteq G \). Then \( HG_\Phi = G \) and so \( G/G_\Phi \simeq H/(H \cap G_\Phi) \subseteq \Phi \) whence \( H_\Phi \subseteq G_\Phi \). Since \( H_\Phi \) is now forced to be abelian and \( F \) is an \( \Phi \)-subgroup of \( H, FC \subseteq H_\Phi \neq E \) by induction on \( H \). On the other hand the fact that \( G_\Phi \) is abelian implies \( H_\Phi \) is normal in \( G \) and hence \( H_\Phi \cap F \) contains \( N_\Phi \), a contradiction. Thus \( F \) is maximal in \( G \) and \( G_\Phi /N_\Phi \) and \( N_\Phi \) are successive chief factors of \( G \). From the uniqueness of \( N_\Phi \), \( G_\Phi \) is an abelian \( \Phi \)-group.

Let \( Q \) be a \( \Phi' \)-subgroup of \( G \) such that \( QG_\Phi \) is normal in \( G \). Then, since \( G_\Phi \) is abelian, \( Q = C_{G_\Phi}(Q) \times [Q, G_\Phi] \) where each component is normal in \( G \). Suppose \( [Q, G_\Phi] \neq G_\Phi \). Then uniqueness of \( N_\Phi \) implies \( C_{G_\Phi}(Q) = G_\Phi \) and \( Q \) is then normal in \( G \). Because of the uniqueness of \( N_\Phi \), \( Q = E \). Thus if \( QG_\Phi \Delta G \) either \( Q = E \) or \( G_\Phi \neq [Q, G_\Phi] \).

Choose \( B_\Phi \) so that \( B_\Phi /N_\Phi = Q_\Phi(G/N_\Phi) \), and set \( T_\Phi = O_{\Phi'\Phi}(F) \). We shall show that \( T_\Phi \subseteq B_\Phi \).

Suppose \( \Phi \neq \Phi' \). Then \( G_\Phi \subseteq O_{\Phi'}(G) \), and \( T_\Phi G_\Phi \) is \( \Phi \)-nilpotent and normal in \( G \). Hence \( T_\Phi G_\Phi \subseteq O_{\Phi'\Phi}(G) \subseteq B_\Phi \).

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Suppose $q = p$. Set $Q_0 = O_{p'}(F)$. Since $Q_0G_{\mathfrak{F}}$ is normalized by both $F$ and $G_{\mathfrak{F}}$, it is normal in $G$. By a previous remark, if $Q_0 \neq E$, $G_{\mathfrak{F}} = [Q_0, G_{\mathfrak{F}}]$. But the latter is impossible since $Q_0 \subseteq O_{p'}(F)$ and $N_0 \subseteq O_p(F)$ imply $[Q, N_0] = E$. Thus $Q_0 = E$. As a result, $T_p = O_p(F)$ and $T_pG_{\mathfrak{F}}$, being normalized by $F$ and $G_{\mathfrak{F}}$, lies in $O_p(G) \subseteq B_p$. Hence $T_p \subseteq B_p$.

Since $\mathfrak{F}$ is saturated, we may assume $\mathfrak{F}$ is locally defined by $\{f(p)\}$. Thus $F \subseteq \mathfrak{F}$ implies $F/T_q \subseteq f(q)$ for each prime $q$ dividing $|F|$. Since $T_q \subseteq B_q$, it follows that $G/B_q \cong F/(F \cap B_q)$ is a homomorphic image of $F/T_q$. Thus $G/B_q \subseteq f(q)$ for each prime, $q$, dividing $[G:N_0]$. Since $\mathfrak{F}$ is locally defined by $\{f(q)\}$, $G/N_0 \subseteq \mathfrak{F}$ whence $G_{\mathfrak{F}} \subseteq N_0$, a contradiction, and (i) is proved.

(ii) In proving the second part of the theorem it suffices to show that every complement, $K$, of $G_{\mathfrak{F}}$ in $G$, is an $\mathfrak{F}$-subgroup of $G$. Again, there is nothing to prove if $G_{\mathfrak{F}} = E$. We suppose that $G_{\mathfrak{F}} \neq E$. Let $K$ be an arbitrary complement of $G_{\mathfrak{F}}$ in $G$ and choose $N$ minimal normal in $G$ contained in $G_{\mathfrak{F}}$. Then $KN \cap G_{\mathfrak{F}} = (K \cap G_{\mathfrak{F}})N = N$ so $KN/N$ is a complement of $G_{\mathfrak{F}}/N = (G/N)_{\mathfrak{F}}$ in $G/N$. By induction $KN/N$ is an $\mathfrak{F}$-subgroup of $G/N$ and so $KN = FN$ where $F$ is an $\mathfrak{F}$-subgroup of $G$.

Suppose $N \subseteq G_{\mathfrak{F}}$. Then $KN = FN \subseteq G$. Now from (i), $F \cap N \subseteq F \cap G_{\mathfrak{F}} = E$ and so $F \subseteq FN$. Consequently, $(FN)_{\mathfrak{F}}$, being a nontrivial characteristic subgroup of $N$ must coincide with $N$. Since $K$ complements $(KN)_{\mathfrak{F}} = N$ in $KN$, induction on $KN$ yields that $K$ is an $\mathfrak{F}$-subgroup of $KN$. Since $F$ is an $\mathfrak{F}$-subgroup of $KN$ as well as $G$, $K$ and $F$ are conjugate in $KN$. Thus $K$ is an $\mathfrak{F}$-subgroup of $G$.

Suppose $N = G_{\mathfrak{F}}$. Then $K \subseteq \mathfrak{F}$ and $K$ is maximal in $G$. Under these circumstances $K$ satisfies the defining properties of an $\mathfrak{F}$-subgroup of $G$.

References


Southern Illinois University