EXISTENCE OF LOCAL CROSS-SECTIONS IN LINEAR CARTAN G-SPACES UNDER THE ACTION OF NONCOMPACT GROUPS

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1. **Introduction and summary.** Let G be a transformation group acting on a space X. If G is a compact Lie group, the existence of a *slice* at every point of X has been shown with increasing generality by Koszul [**3**], Montgomery and Yang [**4**] and Mostow [**6**] (definitions of various italicized terms will be given in §2). A somewhat more restricted, but for certain purposes more useful, object is a *local cross-section* at a point of X. Its existence under some conditions was shown by Gleason [**2**] if G is a compact Lie group. Things become much more difficult if G is not assumed to be compact. It was shown by Palais [**7**] that if G is an arbitrary Lie group, with compact isotropy group in each $x \in X$, the existence of a slice at each x is equivalent to X being a *Cartan G-space.*

In this paper we shall restrict X to an open subset of n-dimensional Euclidean space, G a Lie group of linear transformations of X onto X. If, in addition, X is a Cartan G-space, we shall show that it is possible to remove from X an invariant set of Lebesgue measure 0, such that at every point of the remaining open set X^0 there is a local cross-section. All orbits in X^0 are of maximum dimension.

A local cross-section at x makes a neighborhood of the orbit of x into a product space. This allows integration over the neighborhood to be carried out first "along" and then "across" the orbits. As an application in statistics we mention the problem of definition and computation of the probability density of orbits.

2. **Slices and cross-sections.** The terminology and notation of Palais in [**7**] will be followed closely. We shall define a *linear G-space* to be a couple (X, G), where X is an open subset of Euclidean n-space E^n and G is a Lie group of linear transformations of X onto itself (this is slightly more restricted than the definition in [**7**]). When G is held fixed, X alone will also be called a linear G-space. Most definitions and lemmas in this section are equally valid for G-spaces that are not

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necessarily linear (for definition see [7]). The image of $x \in X$ under $g \in G$ will be written gx. Given any basis in E^n, G can be considered a subgroup of the group $GL(n, R)$ of all $n \times n$ real nonsingular matrices. The *orbit* of x is $Gx = \{gx: g \in G\}$. More generally, for any $S \subset X$, $GS = \{gS: g \in G\}$. The *isotropy group* of x is $G_x = \{g \in G: gx = x\}$. There is a 1-1 correspondence between Gx and $G/G_x = \{gG_x: g \in G\}$ = the homogeneous space of left cosets of G modulo G_x. If X and Y are any two spaces on which G acts, a map $f: X \to Y$ is called *equivariant* if $f(gx) = gf(x)$ for all x, g.

A *slice* at x is a set $S \subset X$ such that (i) $x \in S$; (ii) GS is open in X; (iii) there exists an equivariant map $f: GS \to G/G_x$ such that $f^{-1}(G_x) = S$. For future reference we state the following lemma, which is well known [7] and an immediate consequence of the equivariance of f:

LEMMA 1. *If S is a slice at x, then $G_s \subset G_x$ for every $s \in S$.*

A *local cross-section* at x is a slice S at x such that if s and gs are both in S then $gs = s$. Thus, every orbit intersects S in at most one point. *A global cross-section* at x is a local cross-section S at x such that $GS = X$.

In the theory of Lie groups the notion of a local cross-section is familiar. If G is a Lie group, H a closed subgroup, then a local cross-section in G/H is an open neighborhood U of H in G/H together with an analytic map $\chi: U \to G$ such that $\chi(H) = e$ and $\phi\chi$ = identity map, where ϕ is the natural map $G \to G/H$. Then $\chi(U)$ is a local cross-section at e as defined for transformation groups if we consider G as an H-space, the action of H on G being right multiplication.

The lemma stated below is a specialization of Proposition 2.1.2 of Palais [7].

LEMMA 2 (PALAIS). *Let S be a slice at x and let $\chi: U \to G$ be a local cross-section in G/G_x. Then the map $(u, s) \to \chi(u)s$ is a homeomorphism of $U \times S$ onto an open neighborhood of S in GS.*

We come now to an important characterization of a local cross-section as opposed to merely a slice.

LEMMA 3. *Let S be a slice at x, then S is a local cross-section at x iff $G_s = G_x$ for every $s \in S$.*

PROOF. Let $s \in S$. As a consequence of Lemma 1 we need only show that S is a local cross-section at x iff $G_x \subset G_s$. First observe that $gs \in S$ iff $g \in G_x$, using the equivariance of f defined by S. Then (S is a local cross-section at x) iff $(gs \in S \Rightarrow gs = s)$ iff $(g \in G_x \Rightarrow g \in G_s)$.

If S is a local cross-section at x it follows immediately from Lemma

3 that the map $(gG_x, s)\to gs$ of $G/G_x \times S \to GS$ is well defined and 1-1. That it is a homeomorphism follows from Lemma 2. For certain purposes it may be desirable to have S a differentiable manifold. It turns out that in our linear G-space, if there is a local cross-section at x at all, it can be chosen analytic, even "flat." We shall call a k-dimensional submanifold S of E^n *flat* if S is contained in a translate of a k-dimensional subspace of E^n. In the following we shall restrict ourselves to flat slices and cross-sections. If S is a flat local cross-section at x, the homeomorphism $(gG_x, s)\to gs$ is analytic.

Palais [7] introduced the following concepts: If U, V are subsets of X, denote $((U, V)) = \{g \in G: gU \cap V \neq \emptyset\}$. V is called *thin* if $((V, V))$ has compact closure. X is called a *Cartan G-space* if every $x \in X$ has a thin neighborhood. If x has a thin neighborhood, G_x is necessarily compact. For future reference we quote, below, a result of Palais, which is really a contraction of his Proposition 2.1.7 and his Lemma 2.2 in [7], stated for a linear G-space and slightly modified (by changing differentiability to flatness).

LEMMA 4 (PALAIS). *Let X be a linear Cartan G-space, $x \in X$ and S^* a flat submanifold of X containing x and invariant under G_x. Denote by S^*_x and $(Gx)_x$ the tangent spaces at x to S^* and Gx, respectively, and suppose that S^*_x is a linear complement to $(Gx)_x$ in E^n. Then there is a neighborhood S of x in S^* which is a flat slice at x.*

The question remains under what conditions is a linear G-space a Cartan G-space. The compactness of G_x at each x is, of course, necessary, but it is not sufficient as shown by the following counter example. Let $n=4$, and write $E^4 = E_1 \times E_2$ where E_1 and E_2 are both copies of E^2. Let X consist of those points of E^4 neither of whose projection on E_i, $i=1, 2$, vanishes. Let G be the additive group of reals acting on E^4 as follows: for any $-\infty < t < \infty$, E_1 is rotated over an angle t, E_2 over an angle at, where a is irrational. Then, for every $x \in X$, G_x is trivial, but the orbit of x keeps passing arbitrarily close to x for arbitrarily large t. Thus, X is not Cartan, and, moreover, there is no local cross-section at any x. In this example G is not closed in $GL(n, R)$. If G is closed and every G_x compact it seems to be an open question whether every linear G-space is a Cartan G-space.

3. **Lower-dimensional orbits.** We shall use μ_n to denote n-dimensional Lebesgue measure. Let $m = \max_{x \in X} \dim Gx$. If x lies on an orbit of lower dimension, say $\dim Gx = d < m$, then there can be no local cross-section S at x. For, suppose the contrary, then $G_s = G_x$ for all $s \in S$, by Lemma 3, and it follows that the orbit dimension

would be $=d$ throughout the open set GS, contradicting the conclusion of

THEOREM 1. *Let* $N=\{x\in X: \dim Gx<m=\max_{y\in X}\dim Gy\}$. *Then* N *is an invariant set, closed in* X, *and of* μ_n *measure* 0.

PROOF. In the proof we may assume $\dim G>0$ and $m>0$. Let $\mathfrak{A}$ be the Lie algebra of G, considered as an algebra of $n\times n$ matrices. For any $x\in X$ denote by $(Gx)_x$ the tangent space at x to Gx, then $(Gx)_x=\{Ax: A\in\mathfrak{A}\}=\mathfrak{A}x$, where $x=(x_1, \cdots, x_n)'$ is considered a column vector. Let $A_1, \cdots, A_r$ be a basis for $\mathfrak{A}$ and denote by $L(x)$ the $n\times r$ matrix whose ith column is A_ix. Then $(Gx)_x$ is spanned by the columns of $L(x)$ so that $\dim(Gx)_x=\operatorname{rank} L(x)$. Now $\operatorname{rank} L(x)<m$ for those x for which all $m\times m$ determinants from $L(x)$ are 0. Since these determinants are continuous functions of x, it follows that $N=\{x: \operatorname{rank} L(x)<m\}$ is closed in X. That N is invariant is obvious from its definition. It remains to show that $\mu_nN=0$.

Let $y\in X$ have $\dim Gy=m$, so that $y\neq 0$. By making a nonsingular linear transformation in E^n we may suppose $y=(1, 0, \cdots, 0)'$. There exist $A_1, \cdots, A_m\in\mathfrak{A}$ such that $A_1y, \cdots, A_my$ are linearly independent, i.e. the first columns of $A_1, \cdots, A_m$ are linearly independent. Let $K(x)$ be the $n\times m$ matrix whose ith column is A_ix and set $M=\{x: \operatorname{rank} K(x)<m\}$, then $N\subset M$. Premultiplying the A_i by a nonsingular C premultiplies $K(x)$ by C but leaves M unchanged. We shall keep the same symbols for the new matrices. Choose C so that, for $i=1, \cdots, m$, A_i has first column consisting of zeros except a 1 in the ith row. Let $K_1(x)$ be the submatrix of $K(x)$ formed by the latter's first m rows. Due to the form of the A_i, x_1 appears in $K_1(x)$ only in its diagonal elements, and in each of them with coefficient 1. Let $P(x)$ be the determinant of $K_1(x)$, then $P(x)$ is a polynomial in $x_1, \cdots, x_n$ in which x_1^m appears with coefficient 1. Therefore, $P(x)$ is not the zero polynomial and it follows that $\mu_n\{x: P(x)=0\}=0$. We have now $N\subset M\subset\{x: P(x)=0\}$ so that $\mu_nN=0$, concluding the proof of the theorem.

G, considered as a group of matrices, acts on E^n. By writing Theorem 1 for E^n instead of for X, and observing that X is open in E^n and therefore of positive μ_n measure, we have the immediate

COROLLARY. $\operatorname{Max}_{x\in X}\dim Gx=\max_{x\in E^n}\dim Gx$.

As a consequence of the corollary, the maximum orbit-dimension is independent of X as long as X is open.

4. Existence of local cross-sections. Even if x is on an orbit of maximum dimension, it is not true in general that there is a local cross-section at x, as the following example will show. Let X be E^2 minus the origin and let G consist of all 2×2 matrices of the form $\operatorname{diag}(c, \pm c)$, $c>0$, then all orbits are one-dimensional. If x has $x_2\neq 0$, G_x is trivial, but if $x_2=0$, G_x consists of the identity matrix and $\operatorname{diag}(1, -1)$. Using Lemma 3 it follows that no x with $x_2=0$ can have a local cross-section (a slice S at such an x will intersect every Gs, $s\in S$, $s\neq x$, in 2 points). This example illustrates that necessary for the existence of a local cross-section at x is not only the minimality of dim G_x but also the number of components of G_x. Compare Montgomery and Zippin [**5**, Remark, p. 222] (for compact G). If in the example we remove the set $\{x: x_2=0\}$, which is of μ_n measure 0, then there is a local cross-section at every point of the remainder of the space. In Theorem 2 we shall show that this can be done in general if X is a linear Cartan G-space.

LEMMA 5. *If X is a linear Cartan G-space then at each $x\in X$ there is a flat slice S and a subset S^0 of S, relatively open in S and with $\mu_k(S-S^0)=0$ ($k=\dim S$) such that at each $z\in S^0$ there is a flat local cross-section $S(z)\subset S^0$.*

PROOF. I_r denotes the $r\times r$ identity matrix, Ω_r stands generically for an $r\times r$ orthogonal matrix. Take $x\in X$ arbitrarily. With the same notation as in the proof of Theorem 1 we have $(Gx)_x=\mathfrak{A}x$. Observe that G_x leaves $\mathfrak{A}x$ invariant. Hypotheses and conclusion of the lemma are unchanged if E^n is subjected to a nonsingular linear transformation. Since G_x is compact, we can choose the transformation so that G_x becomes orthogonal [**1**, Theorem 1, p. 176]. Denote $\mathfrak{A}x$ by E_2 and its orthogonal complement in E^n by E_1 and take an orthogonal basis for E^n such that the first k basis vectors span E_1. Then the matrices of G_x have the form $\operatorname{diag}(\Omega_k, \Omega_{n-k})$. Let G_x^0 be the subgroup of G_x whose matrices are of the form $\operatorname{diag}(I_k, \Omega_{n-k})$.

The flat manifold $S^*=x+E_1$ satisfies the conditions of Lemma 4 with $S_x^*=E_1$ and $(Gx)_x=E_2$. By the conclusion of Lemma 4 there is at x a flat slice $S\subset S^*$ (the existence of a slice at x was, of course, already proved by Palais [**7**] Theorem 2.3.3). Take any $s\in S$. By Lemma 1, $G_s\subset G_x$. Let $G_s^0=G_s\cap G_x^0$. Take any $g\in G_x^0$ and observe $g(s-x)=s-x$, since $s-x\in E_1$. Combining this with $gx=x$ and using the linearity of g, we find $gs=s$, so $g\in G_s$. Therefore, $G_x^0\subset G_s$, and from the definition of G_s^0 it follows that $G_s^0=G_x^0$.

Let H_x be the group of matrices Ω_k appearing in the upper left corner of the matrices $\operatorname{diag}(\Omega_k, \Omega_{n-k})$ of G_x. H_x may be considered to act on E_1. Define S^0 to be the relatively open subset of S with the property that $s\in S^0$ implies that $s-x$ has a nonzero projection into each of the irreducible subspaces of E_1 under H_x. Then the only matrix in H_x that leaves $s-x$ invariant is I_k. Consequently, if $s\in S^0$ then $G_s=G_s^0=G_x^0$. It follows readily from the construction that $\mu_k(S-S^0)=0$.

Take any $z\in S^0$. Denote $(Gz)_z$ by $E_2(z)$ and its orthogonal complement by $E_1(z)$. Then $G_z=G_x^0\subset G_x$ implies $E_1(z)\subset E_1$. Analogous to the construction of S at x there is at z a flat slice $S(z)\subset z+E_1(z)\subset z+E_1 =x+E_1$. Since S^0 is open in $x+E_1$ and contains z, we may choose $S(z)$ so small that $S(z)\subset S^0$. Now for every $s\in S^0$, $G_s=G_x^0=G_z$, and it follows then from Lemma 3 that $S(z)$ is a flat local cross-section at z. This concludes the proof of the lemma.

THEOREM 2. *If X is a linear Cartan G-space there exists an invariant set $X^0\subset X$, open in E^n and with $\mu_n(X-X^0)=0$, such that at every $x\in X^0$ there is a flat local cross-section.*

PROOF. To each $x\in X$ choose S_x and S_x^0 according to Lemma 5. According to Lemma 2 there is a homeomorphism of $U\times S_x$ onto an open neighborhood V_x of S_x. Let V_x^0 be the image of $U\times S_x^0$ under this homeomorphism. Since $\mu_k(S_x-S_x^0)=0$, therefore $\mu_n[U\times(S_x-S_x^0)]=0$, and zero measure is preserved under the homeomorphism, it follows that $\mu_n(V_x-V_x^0)=0$. We have $GS_x=\bigcup_{g\in G} gV_x$, $GS_x^0=\bigcup_{g\in G} gV_x^0$. Since X is separable, a countable union suffices: $GS_x=\bigcup_i g_iV_x$, $GS_x^0 =\bigcup_i g_iV_x^0$. Then $GS_x-GS_x^0\subset\bigcup_i g_i(V_x-V_x^0)$ which has μ_n measure 0. Note that GS_x^0 is open. Furthermore, $X=\bigcup_{x\in X} GS_x$. Define $X^0 =\bigcup_{x\in X} GS_x^0$, then X^0 is open, invariant, and, of course, still a linear Cartan G-space. Again, by separability, we can cover with countable unions, and it follows as before that $\mu_n(X-X^0)=0$. Take $y\in X^0$ arbitrarily, then $y\in GS_x^0$ for some x, i.e. for some $x\in X$, $g\in G$ and $z\in S_x^0$ we have $y=gz$. By the conclusion of Lemma 5 there is a flat local cross-section $S(z)$ at z. Then $gS(z)$ is a flat local cross-section at y, and the proof of the theorem is completed.

Note that every point x of X^0 has the following property: x has a neighborhood V such that $y\in V$ implies that G_y and G_x are conjugate. It is easy to see that in a linear Cartan G-space this property of a point x is a necessary and sufficient condition that there be a local cross-section at x. Compare [2, Theorem 3.6] (for compact G). It also follows from this and Theorem 1 that every $x\in X^0$ has its orbit of maximum dimension.

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