HADAMARD MATRICES OF ORDER CUBE PLUS ONE

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1. **Result.** Let A be an Hadamard design of type 1, and let (X, Y, Z) denote the direct product of matrices X, Y and Z (the direction of the product is unimportant here). Later we shall show that

$$\begin{aligned} B = {} & (I, A, J) + (J, I, A) + (A, J, I) \\ & + (A, A, A) + (A, A^T, A^T) + (A^T, A, A^T) + (A^T, A^T, A) \end{aligned}$$

is also an Hadamard design of type 1. This construction will prove the theorem:

THEOREM. *If there is an Hadamard matrix of type* 1 *and order* h, *then there is an Hadamard matrix of type* 1 *and order* $(h-1)^3+1$.

Williamson [2] shows that there exist Hadamard matrices of type 1 for all orders

$$(1) \qquad 2^a(p_1^{a_1}+1)\cdots(p_r^{a_r}+1)\Big\} \quad \begin{array}{l} a, r = 0, 1, 2, \cdots, \\ a_1, \cdots, a_r = 1, 3, 5, \cdots, \end{array}$$

where each p_i is a prime congruent to 3 modulo 4.

For example, an Hadamard matrix of type 1 and order 16 exists. By our theorem, one also exists of order $15^3+1=16\cdot 211$, which is not one of the numbers (1). However, another construction of Williamson [2] yields an Hadamard matrix (not of type 1) for this order. The first "new" order is 39^3+1.

2. **Definitions and proof.** Throughout this paper I and J denote the identity matrix and the matrix with 1 in every position respectively, of the order required by the context. An a, b matrix is one in which each element is either a or b.

An Hadamard matrix is a 1, -1 matrix H of order h such that $HH^T=hI$. (Necessarily either $h=2$ or h is divisible by 4.) It is of type 1 if $H+H^T=2I$.

An Hadamard design A is a 0, 1 matrix of order $h-1$ such that $AA^T=A^TA=(h/4)I+(h/4-1)J$. (Necessarily $AJ=JA=(h/2-1)J$.) It is of type 1 if $A+A^T=J-I$.

If H is an Hadamard matrix it can be multiplied by generalized permutation matrices to bring it into the form

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$$\begin{bmatrix} 1 & 1 \ldots 1 \\ -1 & \\ \vdots & J - 2A \\ -1 & \end{bmatrix},$$

where A is an Hadamard design. Then H is of type 1 if and only if A is of type 1.

To prove that B is a 0, 1 matrix we write it in the form

$$B = (I, X) + (A, Y) + (A^T, Z)$$

where (P, Q) denotes the direct product of P and Q, and

$$\begin{aligned} X &= (I, A) + (A, J), \\ Y &= (I, I + A) + (A, I + A) + (A^T, I + A^T), \\ Z &= (I, A) + (A, A^T) + (A^T, A). \end{aligned}$$

Since I, A and A^T are mutually disjoint we need only show that X, Y and Z are 0, 1 matrices. And the same reasoning, applied to each, confirms this.

To prove that $B+B^T=J-I$ we need only note that

$$X + X^T = J - I \quad \text{and} \quad Y + Z^T = Y^T + Z = J.$$

It remains to prove that BB^T is a linear combination of I and J. This is straightforward algebraic manipulation. First

$$BB^T = (I, U) + (A, V + (n - 1)W) + (A^T, V^T + (n - 1)W)$$

where

$$\begin{aligned} U &= XX^T + (2n - 1)(YY^T + ZZ^T), \\ V &= XZ^T + YX^T + ZY^T, \\ W &= (Y + Z)(Y + Z)^T \end{aligned}$$

and $n=h/4$. Evaluating U, V and W we obtain

$$\begin{aligned} U &= mI + (m - 1)J, \\ V &= -(n - 1)(4n - 1)I + 6n(2n - 1)J + (n - 1)(I, J), \\ W &= (4n - 1)I + (4n - 1)^2J - (I, J), \end{aligned}$$

where $m=((h-1)^3+1)/4$. It follows that

$$BB^T = mI + (m - 1)J.$$

This completes the proof of the theorem.

It is clear that three different Hadamard designs of type 1 of the same order can be used in constructing B. However, all attempts to apply this method using designs of different orders, have failed.

References

1. A. T. Butson, *Generalized Hadamard matrices*, Proc. Amer. Math. Soc. **13** (1962), 894–898.

2. J. Williamson, *Hadamard's determinant theorem and the sum of four squares*, Duke Math. J. **11** (1944), 65–81.

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