HADAMARD MATRICES OF ORDER CUBE PLUS ONE

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1. Result. Let $A$ be an Hadamard design of type 1, and let $(X, Y, Z)$ denote the direct product of matrices $X$, $Y$ and $Z$ (the direction of the product is unimportant here). Later we shall show that

$$B = (I, A, J) + (J, I, A) + (A, J, I)$$


is also an Hadamard design of type 1. This construction will prove the theorem:

**Theorem.** If there is an Hadamard matrix of type 1 and order $h$, then there is an Hadamard matrix of type 1 and order $(h-1)^3 + 1$.

Williamson [2] shows that there exist Hadamard matrices of type 1 for all orders

$$a_1^a, \ldots, a_r^a = 0, 1, 2, \ldots,$$

where each $p_i$ is a prime congruent to 3 modulo 4.

For example, an Hadamard matrix of type 1 and order 16 exists. By our theorem, one also exists of order $15^3 + 1 = 16 \cdot 211$, which is not one of the numbers (1). However, another construction of Williamson [2] yields an Hadamard matrix (not of type 1) for this order. The first “new” order is $39^3 + 1$.

2. Definitions and proof. Throughout this paper $I$ and $J$ denote the identity matrix and the matrix with 1 in every position respectively, of the order required by the context. An $a$, $b$ matrix is one in which each element is either $a$ or $b$.

An Hadamard matrix is a 1, $-1$ matrix $H$ of order $h$ such that $HHT = hI$. (Necessarily either $h = 2$ or $h$ is divisible by 4.) It is of type 1 if $H + HT = 2I$.

An Hadamard design $A$ is a 0, $1$ matrix of order $h-1$ such that $AA^T = A^TA = (h/4)I + (h/4-1)J$. (Necessarily $AJ = JA = (h/2-1)J$.) It is of type 1 if $A + AT = J - I$.

If $H$ is an Hadamard matrix it can be multiplied by generalized permutation matrices to bring it into the form

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\[
\begin{pmatrix}
1 & 1 & \ldots & 1 \\
-1 & & & \\
& \vdots & \ddots & \vdots \\
& & & J - 2A \\
-1 & & & 
\end{pmatrix},
\]

where \( A \) is an Hadamard design. Then \( H \) is of type 1 if and only if \( A \) is of type 1.

To prove that \( B \) is a 0, 1 matrix we write it in the form

\[
B = (I, X) + (A, Y) + (A^T, Z)
\]

where \((P, Q)\) denotes the direct product of \( P \) and \( Q \), and

\[
\begin{align*}
X &= (I, A) + (A, J), \\
Y &= (I, I + A) + (A, I + A) + (A^T, I + A), \\
\end{align*}
\]

Since \( I, A \) and \( A^T \) are mutually disjoint we need only show that \( X, Y \) and \( Z \) are 0, 1 matrices. And the same reasoning, applied to each, confirms this.

To prove that \( B + B^T = J - I \) we need only note that

\[
X + X^T = J - I \quad \text{and} \quad Y + Z^T = Y^T + Z = J.
\]

It remains to prove that \( BB^T \) is a linear combination of \( I \) and \( J \). This is straightforward algebraic manipulation. First

\[
BB^T = (I, U) + (A, V + (n - 1)W) + (A^T, V^T + (n - 1)W)
\]

where

\[
\begin{align*}
U &= XX^T + (2n - 1)(YY^T + ZZ^T), \\
V &= XZ^T + YX^T + ZY^T, \\
W &= (Y + Z)(Y + Z)^T
\end{align*}
\]

and \( n = \frac{h}{4} \). Evaluating \( U, V \) and \( W \) we obtain

\[
\begin{align*}
U &= mI + (m - 1)J, \\
V &= - (n - 1)(4n - 1)I + 6n(2n - 1)J + (n - 1)(I, J), \\
W &= (4n - 1)I + (4n - 1)^2J - (I, J),
\end{align*}
\]

where \( m = ((h - 1)^3 + 1)/4 \). It follows that

\[
BB^T = mI + (m - 1)J.
\]

This completes the proof of the theorem.
It is clear that three different Hadamard designs of type 1 of the same order can be used in constructing $B$. However, all attempts to apply this method using designs of different orders, have failed.

References


National Bureau of Standards