A PROBLEM ON ENDOMORPHISMS OF PRIMARY ABELIAN GROUPS

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In this note, an example is constructed which gives a negative answer to the following question posed by R. S. Pierce [2, p. 367].

Let B be a basic p-group, and $\overline{B}$ the torsion completion of B. Let P be a subgroup of the socle of $\overline{B}$ such that $B[p] \subseteq P$. Let α be an endomorphism of $\overline{B}$ such that $\alpha(P) \subseteq P$. Does there exist a pure subgroup G of $\overline{B}$ such that $B \subseteq G$, $G[p] = P$ and $\alpha(G) \subseteq G$?

Let N be the set of positive integers and let $B = \sum_{i \in N} \oplus B_i$ be a standard basic group with projections $\rho_i \colon B \to B_i$. Note that $B_i = \{b_i\}$ is cyclic and of order p^i. For notational convenience, let $c_i = p^{i-1} b_i$. Let $y = \sum_{i \in N} c_{3i}$ and let P be the subgroup of $\overline{B}[p]$ generated by $B[p]$ and $\{y\}$. Let α be the endomorphism of $\overline{B}$ determined by the conditions:

$$\alpha(b_i) = \begin{cases} b_i + p^2 b_{i+1} & \text{if } i = 3n \\ p b_{i+1} & \text{if } i = 3n+1 \\ b_i & \text{if } i = 3n+2. \end{cases}$$

It follows that $\alpha(y) = y$ and that $\alpha(B) \subseteq B$. Thus,

$$\alpha(P) = \alpha(B[p] + \{y\}) \subseteq \alpha(B[p]) + \alpha(\{y\}) \subseteq B[p] + \{y\} = P.$$

Suppose there is a pure subgroup G of $\overline{B}$ such that $B \subseteq G$, $G[p] = P$ and $\alpha(G) \subseteq G$. Let $x = \sum_{i \in N;\, i > 2} a_i b_i \in G$ be such that $px = y$. There must be such an element x since $B \subseteq G$, G is pure in $\overline{B}$ and since the height of y in $\overline{B}$ is 2. Now for each $i \in N$, $\rho_{3i}(\alpha(x) - x) = 0$ since $\rho_{3i}(x) = a_{3i} b_{3i} = \rho_{3i} \alpha(a_{3i} b_{3i}) = \rho_{3i} \alpha(x)$. Also, $\alpha(x) - x \in P$ since $p(\alpha(x) - x) = \alpha(y) - y = 0$. Consequently, by the definition of P, $\alpha(x) - x \in B$. It follows that $\rho_i(\alpha(x) - x) = 0$ for all but a finite number of indices $i \in N$. Now, since

$$\alpha(x) = \sum_{i \in N} a_{3i} \alpha(b_{3i}) + \sum_{i \in N} a_{3i+1} \alpha(b_{3i+1}) + \sum_{i \in N} a_{3i+2} \alpha(b_{3i+2})$$

$$= \sum_{i \in N} (a_{3i} b_{3i} + a_{3i} p^2 b_{3i+1} + a_{3i+1} p b_{3i+2} + a_{3i+2} b_{3i+2}),$$

$$\alpha(x) - x = \sum_{i \in N} (a_{3i} p^2 - a_{3i+1}) b_{3i+1} + \sum_{i \in N} a_{3i+1} p b_{3i+2}.$$

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Therefore

$$\rho_{3i+2}(\alpha(x) - x) = a_{3i+1}pb_{3i+2}$$

and

$$\rho_{3i+1}(\alpha(x) - x) = (a_{3i}p^2 - a_{3i+1})b_{3i+1}.$$

Thus, if $\rho_i(\alpha(x)-x)=0$ for almost all $i \in N$, then p^{3i+1} divides a_{3i+1} and, consequently, p^{3i+1} divides $a_{3i}p^2$ for almost all indices $i \in N$. This implies that p^{3i-1} divides a_{3i} for almost all i. Thus,

$$\rho_{3i}(y) = p\rho_{3i}(x) = pa_{3i}b_{3i} = 0$$

for almost all $i \in N$, contradicting the definition of y. Therefore, no such pure subgroup G exists.

References

1. L. Fuchs, *Abelian groups*, Publ. House Hungar. Acad. Sci. Budapest, 1958.

2. R. S. Pierce, *Homomorphisms of primary abelian groups*, Topics in Abelian Groups, Chicago, Ill., 1963.

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