GENERALIZED DERIVATIVES AND THE DE LA VALLÉE POUSSIN DERIVATIVE[1]

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Let $V_n[f: p_1, \cdots, p_{n+1}]$ denote the nth divided difference of f relative to the distinct points $p_1, \cdots, p_{n+1}$, where f is a real valued continuous function over the interval (a, b) and $p_i \in (a, b)$, $i = 1, \cdots, n+1$.

If

$$\lim_{h \to 0} n! V_n[f: x + x_1 h, \cdots, x + x_{n+1} h]$$

exists and is finite, it is called, following Denjoy [**1**, p. 305], the *nth E-generalized derivative* of f at x, $f_{n,E}(x)$, where $E = \{x_1, \cdots, x_{n+1}\}$. If $f_{n,E}(x)$ is independent of the choice of $E \subset S$ for a subset S of the reals, we will call it the *nth S-generalized derivative* $f_{n,S}(x)$.

If

$$f(x + t) = a_0 + a_1 t + \cdots + a_n t^n / n! + o(t^n) \tag{1}$$

where the numbers $a_0, a_1, \cdots, a_n$ depend on x only, then a_n will be called, following Marcinkiewicz and Zygmund [**5**, p. 1], the *nth de la Vallée Poussin derivative* of f at x, $f_{(n)}(x)$.

Denjoy has shown [**1**, p. 289] that the existence of $f_{(n)}(x)$ implies that of $f_{n,S}(x)$ for S arbitrary. In a previous paper [**4**] the author has obtained a converse of the above Denjoy theorem. The purpose of the present paper is to establish a more general form of this converse theorem.

Let f and S be as above. Denote by S_{j+1} any $(j+1)$-tuple of distinct points of S, $j = 1, \cdots, n-1$. For $x \in (a, b)$ and $s_i \in S$, set $y_i = x + s_i h$, $i = 1, \cdots, n+1$.

THEOREM. *Suppose that $f_{n,S}(x)$ exists. If*

(i) *$f_{j,S_{j+1}}(x)$ exists for all $j = 1, \cdots, n-1$,*

(ii) $\lim_{h \to 0} \{(y - y_{n+1}) V_{n+1}[f: y_1, \cdots, y_{n+1}, y]\} = \theta(x, y - x)$ *where* $\theta(x, y - x) \to 0$ as $y \to x$, *then $f_{(n)}(x)$ exists and $f_{(n)}(x) = f_{n,S}(x)$.*

PROOF. By [**3**, Corollary 3] we obtain the existence of $f_{1,S}(x), \cdots, f_{n-1,S}(x)$. Consequently, $f_{1,E_2}(x), \cdots, f_{n,E_{n+1}}(x)$ exist, where $E_i = \{s_1, \cdots, s_i\}$ for $i = 1, \cdots, n+1$, and the result follows by [**4**, Theorem].

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REMARKS. (i) As mentioned earlier, Denjoy has shown that the existence of $f_{(n)}(x)$ implies that of $f_{n,S}(x)$ for S arbitrary. Since the existence of $f_{(n)}(x)$ implies by (1) that of $f_{(1)}(x), \cdots, f_{(n-1)}(x)$, it follows that the existence of $f_{(n)}(x)$ implies the first condition of the above theorem.

(ii) The existence of $f_{(n)}(x)$ implies also the second condition of the above theorem. Indeed in this case, the expression $f(x+t)+o(t^n)$ is by (1) a polynomial of degree at most n and it follows by [2, Theorem 2.1] that

$$0 = V_{n+1}[f(x+t)+o(t^n)\colon s_1h, \cdots, s_{n+1}h, y-x].$$

Hence,

$$V_{n+1}[f(t)\colon x+s_1h, \cdots, x+s_{n+1}h, y] = V_{n+1}[o(t^n)\colon s_1h, \cdots, s_{n+1}h, y-x]. \tag{2}$$

Further, evaluating the divided difference we get:

$$\begin{aligned}(y-y_{n+1})&V_{n+1}[o(t^n)\colon s_1h, \cdots, s_{n+1}h, y-x] \\ &= \frac{(y-y_{n+1})o((y-x)^n)}{(y-y_1)\cdots(y-y_{n+1})} \\ &\quad - \sum_{i=1}^{n+1} \frac{(y-y_{n+1})o(s_i^n h^n)}{h^n(y-y_i)\{(u-s_1)\cdots(u-s_{n+1})\}'_{u=s_i}}.\end{aligned}$$

When $h \to 0$, we obtain

$$\lim_{h\to 0}\{(y-y_{n+1})V_{n+1}[o(t^n)\colon s_1h, \cdots, s_{n+1}h, y-x]\} = \theta(x, y-x).$$

The result follows from relation (2).

(iii) The following example shows that the first condition of the above theorem cannot alone ensure the existence of $f_{(n)}(x)$. Consider the continuous function $f(x)=x|x|$. Let $n=2$, $x=0$, $s_1=-1$, $s_2=0$, $s_3=1$. Evaluating the generalized derivatives we find:

$$f_{1,\{-1,0\}}(0) = 0; \qquad f_{2,\{-1,0,1\}}(0) = 0.$$

For $y>0$, we find that

$$\lim_{h\to 0}\{(y-h)V_3[x|x| : -h, 0, h, y]\} = 1 \neq \theta(0, y-0).$$

The de la Vallée Poussin derivative $f_{(2)}(0)$ does not exist because otherwise we must have

$$t|t| = f_{(1)}(0)t + f_{(2)}(0)t^2/2 + o(t^2),$$

whence $f_{(1)}(0) = o(t^2) = 0$. Therefore

$$t|t| = f_{(2)}(0)t^2/2.$$

From this relation we obtain $f_{(2)}(0) = 2$ when $t > 0$, and $f_{(2)}(0) = -2$ when $t < 0$. Since $f_{(2)}(0)$ here depends on t, the second de la Vallée Poussin derivative of $x|x|$ does not exist at the origin.

References

1. A. Denjoy, *Sur l'intégration des coefficients différentiels d'ordre supérieur*, Fund. Math. **25** (1935), 273–326.

2. C. Kassimatis, *Certain properties of the generalized integral of a finite Riemann derivative*, Trans. Roy. Soc. Canad. Ser. III **53** (1959), 31–36.

3. ———, *On generalized derivatives*, Studia Math. **25** (1965), 369–371.

4. ———, *On the de la Vallée Poussin derivative*, Proc. Amer. Math. Soc. **16** (1965), 1171–1172.

5. J. Marcinkiewicz and A. Zygmund, *On the differentiability of functions and summability of trigonometrical series*, Fund. Math. **26** (1936), 1–43.

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