l-SEQUENCES FOR NONEMBEDDABLE FUNCTIONS

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Let Ω be the set of all analytic functions $f(z)$ which are regular for $z=0$ and for which $f(0)=0$ and $f'(0)=1$. Then, for $|z|<\rho$, $\rho>0$:

$$(1) \qquad f(z) = \sum_{k=1}^{k=\infty} f_k z^k, \quad \text{with } f_1 = 1.$$

Consider the problem of embedding a given function $f(z)$ belonging to Ω in a continuous group of $f(s, z)$ (here s is the group parameter and $f(s, z)$ is considered as a function of z) satisfying:

$$(2) \qquad \begin{aligned} &f(1, z) = f(z), \\ &f(s, z) \in \Omega \text{ (qua function of } z\text{) for all real } s, \\ &f[s, f(t, z)] = f[(s+t), z] \text{ for all real } s \text{ and } t. \end{aligned}$$

The function $f(s, z)$ is the s-iterate of $f(z)$. If it exists for all real s (that is if conditions (2) are satisfied), it can be shown to be analytic in s [**3**]. It is then called the *analytic iterate* of $f(z)$.

There exist functions $f(z)$ belonging to Ω which have an analytic iterate. Such, for instance, is the function $f(z)=z/(1-z)$ for which $f(s, z)=z/(1-sz)$. However in 1958 I. N. Baker showed [**1**] that the function $f(z)=e^z-1$, which also belongs to Ω, cannot be embedded in a continuous group $f(s, z)$.

It should be noted that for all functions $f(z)$ belonging to Ω there exists a formal power series:

$$(3) \qquad f(s, z) = \sum_{k=1}^{k=\infty} f_k(s) z^k, \qquad \text{with } f_1(s) = 1,$$

formally satisfying (2) but while for embeddable functions, like $f(z)=z/(1-z)$, the series (3) converges for any given complex s and sufficiently small $|z|>0$, for nonembeddable functions like $f(z)=e^z-1$, the corresponding series, for almost all s [**3**], converges only for $z=0$. However even in this case the series for $f(s, z)$ converges for positive $|z|$ at least for all integer values of s.

It follows that the set Ω splits into two disjoint sets A and B, the set A consisting of all the functions of Ω which are embeddable in a continuous group and the set B of all the nonembeddable ones.

The sequence of coefficients $\{f_k\}$ of $f(z)$ in (1) (note that $f_1=1$)

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generates a sequence of numbers $\{l_k\}$ called the *l-sequence of $f(z)$* [4]. This l-sequence has the property [3] that:

(a) if $f(z)$ belongs to A then the series:

$$\sum_{k=1}^{k=\infty} l_k z^{k+1} = L(z) \tag{4}$$

converges for $|z| < r$, $r > 0$ and represents an analytic function (actually $L(z) = \partial f(s, z)/\partial s|_{s=0}$);

(b) if $f(z)$ belongs to B then the series in (4) has a zero radius of convergence and the function $L(z)$ does not exist.

It is the purpose of this paper to study the sequences $\{l_k\}$ in the case when $f(z)$ belongs to B.

Our results can be summarized in the following two theorems:

THEOREM I. *If $f(z) \in B$ then:*

$$\limsup_{k\to\infty} |l_k|^{1/k} = \infty \tag{5}$$

and:

$$\limsup_{k\to\infty} \frac{1}{k} |l_k|^{1/k} < \infty. \tag{6}$$

THEOREM II. *If $f(z) \in B$ and if all the l_k are real then it is not true that one of the following inequalities holds for all k:*

$$l_k \geqq 0, \ l_k \leqq 0, \ (-1)^k l_k \geqq 0, \ (-1)^k l_k \leqq 0. \tag{7}$$

PROOF. The equation (5) results from the fact that when $f(z)$ belongs to B, the series (4) for $L(z)$ has a zero radius of convergence.

By a previous remark, if j is a positive integer, the j-iterate $f(j, z)$ of $f(z)$ exists.

Let:

$$f(j, z) = \sum_{k=1}^{k=\infty} f_k^{(j)} z^k.$$

In [5] p. 462 the l_k are given by the formula:[1]

$$l_k = \sum_{j=1}^{j=k} \frac{(-1)^j}{j} C_{k,j} f_{k+1}^{(j)}. \tag{8}$$

As this sum has only k terms and as $C_{k,j} \leqq 2^k$, it follows that:

[1] *Remark.* Note that in [5, Equation 3.4, p. 462], there is an obvious misprint: the summation should start from $q=1$ and not from $q=0$.

$$(9) \qquad |l_k| \leqq k\cdot 2^k\cdot \max_{1\leqq j\leqq k} |f_{k+1}^{(j)}|.$$

To estimate $\max_{1\leqq j\leqq k} |f_{k+1}^{(j)}|$ we note that as $f(z)$ belongs to Ω it has a positive radius of convergence. Since, moreover, $f_1=1$, there exists a finite $c>0$ such that $|f_k|\leqq c^{k-1}$. The function $f(z)$ is thus majorized by the function $z/(1-cz)$. It easily follows that the j-iterate of $f(z)$ is majorized by the j-iterate of $z/(1-cz)$, which is $z/(1-jcz)$. Hence

$$|f_{k+1}^{(j)}| \leqq (jc)^k \quad \text{and} \quad \max_{1\leqq j\leqq k} |f_{k+1}^{(j)}| \leqq (kc)^k,$$

so that, finally:

$$|l_k| \leqq k\cdot 2^k\cdot (ck)^k,$$

and:

$$\limsup_{k\to\infty} \frac{1}{k} |l_k|^{1/k} \leqq 2c < \infty,$$

which is (6).

To prove Theorem II we note than, in [5, p. 465] the f_k are given by a formula of the type:

$$f_k = l_{k-1} + P_k(l_1, l_2, \cdots, l_{k-1}),$$

where p_k is a polynomial with nonnegative coefficients. Hence, if all $l_k\geqq 0$, then $f_k\geqq l_{k-1}$ for all k. But the f_k are the coefficients of the power series for $f(z)$ which has a positive radius of convergence so that the series for $L(z)$ has then also a positive radius of convergence, which contradicts the assumption that $f(z)$ belongs to B. Hence not all the l_k are nonnegative.

Next, if $\{l_k\}$ is the l-sequence for $f(z)$, then the sequences $\{-l_k\}$, $\{(-1)^k l_k\}$ and $\{(-1)^{k+1}l_k\}$ are respectively the l-sequences for the functions $f^{-1}(z)$ (the inverse function of $f(z)$), $-f(-z)$ and $-f^{-1}(-z)$. If either of these functions were to belong to A, then $f(z)$ would also belong to A, contrary to our assumption. It follows, as above, that none of the sequences $\{-l_k\}$, $\{(-1)^k l_k\}$ and $\{(-1)^{k+1}l_k\}$ can be nonnegative, which completes the proof.

Theorems I and II show that the l-sequence for nonembeddable functions, even when real, are most unwieldy. Yet many of the usual functions of Ω are nonembeddable. Baker showed this for e^z-1 in 1958 [7]; Lewine, in his M.Sc. Thesis in 1960 [6] [7], showed this for $z+z^2$ and for $z/(1-z)^2$; Szekeres showed in 1964 [8] that all entire functions and all rational functions except $z/(1-cz)$ are nonembeddable. Finally Baker showed this in 1964 [2] for all meromorphic functions (again except $z/(1-cz)$). On the other hand l-sequences appear in the theory of conformal mapping, schlichtness [5] and, possibly, in other problems. The configuration space of these sequences may prove to be important in these problems, yet very little can be said about it. Not a single explicit example of an l-sequence for a nonembeddable function has been exhibited.

Added in proof. It has been brought to our attention that P. C. Rosenbloom proved in *Communications du seminaire mathematique de l'université de Lund, tome supplementaire* (1952) *dedie a Marcel Riesz*, pp. 186–192, that no function of the type $F(z)=z+z^2l^{G(z)}$ ($G(z)$ an entire function) is the iterate of $[f(z)]$ of an entire function $f(z)$. As it can be easily shown that no entire function of Ω is the iterate of a nonentire function of Ω, this seems to be the first proof on record of the existence of function of type B.

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