ON FUNCTIONS SATISFYING $R\{f(z)/z\} > 0$

KUNIO YAMAGUCHI

1. We shall say that $f(z)$ belongs to the class $S_0$, if $f(z)$ is analytic in $|z| < 1$,

$$R\{f(z)/z\} > 0 \quad (|z| < 1),$$

and

$$f(0) = 0, \quad f'(0) = 1.$$

Our principal result is

**Theorem 1.** If $f(z) \in S_0$, then

$$Rf'(re^{i\theta}) \geq \frac{1 - 2r - r^2}{(1 + r)^2} \quad \text{for} \quad 0 \leq r < 2^{1/2} - 1.$$

This bound is sharp.

2. **Proof of Theorem 1.** The function $g(z) = f(z)/z$ has

$$Rg(z) > 0, \quad g(0) = 1.$$

It is therefore subordinate to the function

$$w = \frac{1 - z}{1 + z}$$

which maps $|z| < 1$ on $Rw > 0, w(0) = 1$. Hence [2, p. 356]

$$g(z) = f(z)/z = \frac{1 - k(z)}{1 + k(z)},$$

where $k(z)$ is analytic in $|z| < 1$ and

$$|k(z)| < 1 \quad (|z| < 1), \quad k(0) = 0.$$

Hence

$$f'(z) = \frac{1 - k(z)}{1 + k(z)} - \frac{2zk'(z)}{(1 + k(z))^2}.$$

By Schwarz's Lemma, (2) implies

$$k(z) = z\phi(z), \quad k'(z) = \phi(z) + z\phi'(z),$$

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when $\phi(z)$ is analytic in $|z| < 1$ and
\begin{equation}
|\phi(z)| < 1 \quad (|z| < 1).
\end{equation}

Therefore
\begin{equation*}
f'(z) = \frac{1 - 2k - k^2 - 2z^2\phi'(z)}{(1 + k)^2} = \frac{(1 + k)^2(1 - 2k - k^2 - 2z^2\phi'(z))}{|1 + k|^4}.
\end{equation*}

Taking real parts gives
\begin{equation*}
Rf'(z) = \left\{1 - 4 |k|^2 - 2 |k|^2(k + \bar{k}) - |k|^4 - 2Rz^2\phi'(z)(1 + \bar{k})^2\right\}|1 + k|^{-4}
\end{equation*}
\begin{equation*}
\geq \left\{1 - 4 |k|^2 - 4 |k|^3 - |k|^4 - 2 |z|^2 |\phi'(z)| (1 + |k|)^2\right\}|1 + k|^{-4}.
\end{equation*}

By a well-known extension of Schwarz’s Lemma [1, p. 18], (3) implies
\begin{equation*}
|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}.
\end{equation*}

Therefore, putting $|z| = r$, $|k| = |z\phi(z)| = u$ and noting that
\begin{equation}
1 - 4u^2 - 4u^3 - u^4 = (1 + u)^2(1 - 2u - u^2),
\end{equation}

\begin{equation*}
Rf'(z) \geq \left\{1 - 2u - u^2 + \frac{2u^2}{1 - r^2} - \frac{2r^2}{1 - r^2}\right\}(1 + u)^2
\end{equation*}
\begin{equation*}
= h(u)(1 + u)^2|1 + k|^{-4}.
\end{equation*}

By (3),
\begin{equation}
0 \leq u \leq r.
\end{equation}

If $r \leq 2^{1/2} - 1$, $u \leq r$,
\begin{equation*}
\frac{1}{2}h'(u) = \frac{1 + r^2}{1 - r^2}u - 1 \leq \frac{r}{1 - r^2} - 1
\end{equation*}
\begin{equation*}
\leq (2^{1/2} - 1) \frac{1 + (2^{1/2} - 1)^2}{1 - (2^{1/2} - 1)^2} - 1 = 1 - 2^{1/2} < 0,
\end{equation*}

so that
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\[ h(u) \geq h(r) = 1 - 2r - r^2 \geq 0 \quad (0 < r \leq 2^{1/2} - 1). \]

Therefore, by (4) and (5),

\[ Rf'(z) \geq (1 - 2r - r^2)(1 + u)^2 \left| 1 + k \right|^{-4} \]
\[ \geq (1 - 2r - r^2)(1 + u)^2(1 + u)^{-4} \]
\[ \geq (1 - 2r - r^2)(1 + r)^{-2}. \]

The function

\[ f(z) = \frac{1 - z}{1 + z} \in S_0 \]

has

\[ f'(r) = \{1 - 2r - r^2\}(1 + r)^{-2}, \]

which shows that our result is sharp.

3. Applications. Theorem 1 gives a new proof of

**Theorem 2** (Noshiro, [5]). If \( f(z) \in S_0 \), then \( f(z) \) is univalent in \( |z| < 2^{1/2} - 1 \).

This follows by combining the obvious consequence

\[ Rf'(z) > 0 \quad (|z| < 2^{1/2} - 1) \]

with the well-known

**Wolff-Noshiro Lemma.** If \( f(z) \) is analytic in \( |z| < R \) and \( Rf'(z) > 0 \) \((|z| < R)\), then \( f(z) \) is univalent in \( |z| < R \).

**Theorem 3.** If

\[ f(z) = z + a_2z^2 + a_3z^3 + \cdots \in S_0, \]

then the partial sums

\[ s_n(z) = z + a_2z^2 \cdot \cdots + a_nz^n \quad (n = 2, 3, \cdots) \]

are univalent in \( |z| < 1/4 \).

**Proof.** The function

\[ g(z) = f(z)/z = 1 + a_2z + a_3z^2 \cdots \]

satisfies (1). It is well known that this implies [4, p. 170]

\[ |a_n| \leq 2 \quad (n = 2, 3, \cdots). \]

Therefore, putting \( |z| = r \),
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$$R S_n'(z) = R f'(z) - R \sum_{k=n+1}^{\infty} k a_k z^{k-1}$$

$$\geq R f'(z) - 2 \sum_{k=n+1}^{\infty} k r^{k-1}.$$ 

Summing the series on the right-hand side and using Theorem 1 we obtain that, for $|z| < 2^{1/2} - 1$,

$$R S_n'(z) \geq (1 - 2r - r^2) (1 + r)^{-2} - 2[n + 1 - nr] r^n (1 - r)^{-2}.$$ 

If $r = 1/4 (< 2^{1/2} - 1)$, then

$$(n + 1 - nr) r^n \leq (4 - 3r) r^3 \quad (n = 3, 4, 5 \cdots)$$

and so, for $r = 1/4$,

$$R S_n'(z) \geq (7/25) - (13/72) > 0.$$ 

By the minimum modulus principle applied to the harmonic function $R S_n'(z)$ this implies

$$R S_n'(z) > 0 \quad (|z| \leq 1/4).$$

The result now follows from the Wolff-Noshiro Lemma for $n \geq 3$. For $n = 2$ we have, by (6),

$$R S_2'(z) = R (1 + 2 a_2 z) > 1 - 4 |z| > 0 \quad (|z| < 1/4).$$

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References


University of Nagasaki