

ON FUNCTIONS SATISFYING $R\{f(z)/z\} > 0$

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1. We shall say that $f(z)$ belongs to the class S_0 , if $f(z)$ is analytic in $|z| < 1$,

$$R\{f(z)/z\} > 0 \quad (|z| < 1),$$

and

$$f(0) = 0, \quad f'(0) = 1.$$

Our principal result is

THEOREM 1. *If $f(z) \in S_0$, then*

$$Rf'(re^{i\theta}) \geq \frac{1 - 2r - r^2}{(1 + r)^2} \quad \text{for } 0 \leq r < 2^{1/2} - 1.$$

This bound is sharp.

2. **Proof of Theorem 1.** The function $g(z) = f(z)/z$ has

$$(1) \quad Rg(z) > 0, \quad g(0) = 1.$$

It is therefore subordinate to the function

$$w = \frac{1 - z}{1 + z}$$

which maps $|z| < 1$ on $Rw > 0$, $w(0) = 1$. Hence [2, p. 356]

$$g(z) = f(z)/z = \frac{1 - k(z)}{1 + k(z)},$$

where $k(z)$ is analytic in $|z| < 1$ and

$$(2) \quad |k(z)| < 1 \quad (|z| < 1), \quad k(0) = 0.$$

Hence

$$f'(z) = \frac{1 - k(z)}{1 + k(z)} - \frac{2zk'(z)}{(1 + k(z))^2}.$$

By Schwarz's Lemma, (2) implies

$$k(z) = z\phi(z), \quad k'(z) = \phi(z) + z\phi'(z),$$

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when $\phi(z)$ is analytic in $|z| < 1$ and

$$(3) \quad |\phi(z)| < 1 \quad (|z| < 1).$$

Therefore

$$\begin{aligned} f'(z) &= \frac{1 - 2k - k^2 - 2z^2\phi'(z)}{(1 + k)^2} \\ &= \frac{(1 + \bar{k})^2(1 - 2\bar{k} - \bar{k}^2 - 2z^2\phi'(z))}{|1 + k|^4}. \end{aligned}$$

Taking real parts gives

$$\begin{aligned} \Re f'(z) &= \{1 - 4|k|^2 - 2|k|^2(k + \bar{k}) - |k|^4 \\ &\quad - 2\Re z^2\phi'(z)(1 + \bar{k})^2\} |1 + k|^{-4} \\ &\geq \{1 - 4|k|^2 - 4|k|^3 - |k|^4 \\ &\quad - 2|z|^2|\phi'(z)|(1 + |k|)^2\} |1 + k|^{-4}. \end{aligned}$$

By a well-known extension of Schwarz's Lemma [1, p. 18], (3) implies

$$|\phi'(z)| \leq \frac{1 - |\phi(z)|^2}{1 - |z|^2}.$$

Therefore, putting $|z| = r$, $|k| = |z\phi(z)| = u$ and noting that

$$\begin{aligned} 1 - 4u^2 - 4u^3 - u^4 &= (1 + u)^2(1 - 2u - u^2), \\ (4) \quad \Re f'(z) &\geq \left\{1 - 2u - u^2 + \frac{2u^2}{1 - r^2} - \frac{2r^2}{1 - r^2}\right\} \frac{(1 + u)^2}{|1 + k|^4} \\ &= h(u)(1 + u)^2 |1 + k|^{-4}. \end{aligned}$$

By (3),

$$(5) \quad 0 \leq u \leq r.$$

If $r \leq 2^{1/2} - 1$, $u \leq r$,

$$\begin{aligned} \frac{1}{2}h'(u) &= \frac{1 + r^2}{1 - r^2} u - 1 \\ &\leq r \frac{1 + r^2}{1 - r^2} - 1 \\ &\leq (2^{1/2} - 1) \frac{1 + (2^{1/2} - 1)^2}{1 - (2^{1/2} - 1)^2} - 1 = 1 - 2^{1/2} < 0, \end{aligned}$$

so that

$$h(u) \geq h(r) = 1 - 2r - r^2 \geq 0 \quad (0 < r \leq 2^{1/2} - 1).$$

Therefore, by (4) and (5),

$$\begin{aligned} \Re f'(z) &\geq (1 - 2r - r^2)(1 + u)^2 |1 + k|^{-4} \\ &\geq (1 - 2r - r^2)(1 + u)^2(1 + u)^{-4} \\ &\geq (1 - 2r - r^2)(1 + r)^{-2}. \end{aligned}$$

The function

$$f(z) = z \frac{1 - z}{1 + z} \in S_0$$

has

$$f'(r) = \{1 - 2r - r^2\}(1 + r)^{-2},$$

which shows that our result is sharp.

3. Applications. Theorem 1 gives a new proof of

THEOREM 2 (NOSHIRO, [5]). *If $f(z) \in S_0$, then $f(z)$ is univalent in $|z| < 2^{1/2} - 1$.*

This follows by combining the obvious consequence

$$\Re f'(z) > 0 \quad (|z| < 2^{1/2} - 1)$$

with the well-known

WOLFF-NOSHIRO LEMMA. *If $f(z)$ is analytic in $|z| < R$ and $\Re f'(z) > 0$ ($|z| < R$), then $f(z)$ is univalent in $|z| < R$.*

THEOREM 3. *If*

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S_0,$$

then the partial sums

$$s_n(z) = z + a_2 z^2 + \cdots + a_n z^n \quad (n = 2, 3, \cdots)$$

are univalent in $|z| < 1/4$.

PROOF. The function

$$g(z) = f(z)/z = 1 + a_2 z + a_3 z^2 + \cdots$$

satisfies (1). It is well known that this implies [4, p. 170]

$$(6) \quad |a_n| \leq 2 \quad (n = 2, 3, \cdots).$$

Therefore, putting $|z| = r$,

$$\begin{aligned}
 R s_n'(z) &= R f'(z) - R \sum_{k=n+1}^{\infty} k a_k z^{k-1} \\
 &\geq R f'(z) - 2 \sum_{k=n+1}^{\infty} k r^{k-1}.
 \end{aligned}$$

Summing the series on the right-hand side and using Theorem 1 we obtain that, for $|z| < 2^{1/2} - 1$,

$$R s_n'(z) \geq (1 - 2r - r^2)(1 + r)^{-2} - 2[n + 1 - nr]r^n(1 - r)^{-2}.$$

If $r = 1/4$ ($< 2^{1/2} - 1$), then

$$(n + 1 - nr)r^n \leq (4 - 3r)r^3 \quad (n = 3, 4, 5 \dots)$$

and so, for $r = 1/4$,

$$R s_n'(z) \geq (7/25) - (13/72) > 0.$$

By the minimum modulus principle applied to the harmonic function $R s_n'(z)$ this implies

$$R s_n'(z) > 0 \quad (|z| \leq 1/4).$$

The result now follows from the Wolff-Noshiro Lemma for $n \geq 3$. For $n = 2$ we have, by (6),

$$R s_2'(z) = R(1 + 2a_2z) > 1 - 4|z| > 0 \quad (|z| < 1/4).$$

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