ON FUNCTIONS SATISFYING $R\{f(z)/z\} > 0$

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1. We shall say that $f(z)$ belongs to the class S_0, if $f(z)$ is analytic in $|z| < 1$,

$$R\{f(z)/z\} > 0 \qquad (|z| < 1),$$

and

$$f(0) = 0, \qquad f'(0) = 1.$$

Our principal result is

THEOREM 1. *If $f(z) \in S_0$, then*

$$Rf'(re^{i\theta}) \geqq \frac{1 - 2r - r^2}{(1 + r)^2} \qquad \text{for } 0 \leqq r < 2^{1/2} - 1.$$

This bound is sharp.

2. **Proof of Theorem 1.** The function $g(z) = f(z)/z$ has

$$(1) \qquad Rg(z) > 0, \qquad g(0) = 1.$$

It is therefore subordinate to the function

$$w = \frac{1 - z}{1 + z}$$

which maps $|z| < 1$ on $Rw > 0$, $w(0) = 1$. Hence [2, p. 356]

$$g(z) = f(z)/z = \frac{1 - k(z)}{1 + k(z)},$$

where $k(z)$ is analytic in $|z| < 1$ and

$$(2) \qquad |k(z)| < 1 \qquad (|z| < 1), \qquad k(0) = 0.$$

Hence

$$f'(z) = \frac{1 - k(z)}{1 + k(z)} - \frac{2zk'(z)}{(1 + k(z))^2} \cdot$$

By Schwarz's Lemma, (2) implies

$$k(z) = z\phi(z), \quad k'(z) = \phi(z) + z\phi'(z),$$

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when $\phi(z)$ is analytic in $|z|<1$ and

(3) $$|\phi(z)| < 1 \qquad (|z| < 1).$$

Therefore

$$f'(z) = \frac{1-2k-k^2-2z^2\phi'(z)}{(1+k)^2} = \frac{(1+\bar{k})^2(1-2k-k^2-2z^2\phi'(z))}{|1+k|^4}.$$

Taking real parts gives

$$\begin{aligned} Rf'(z) &= \{1-4|k|^2-2|k|^2(k+\bar{k})-|k|^4 \\ &\qquad - 2Rz^2\phi'(z)(1+\bar{k})^2\}|1+k|^{-4} \\ &\geqq \{1-4|k|^2-4|k|^3-|k|^4 \\ &\qquad - 2|z|^2|\phi'(z)|(1+|k|)^2\}|1+k|^{-4}. \end{aligned}$$

By a well-known extension of Schwarz's Lemma [**1**, p. 18], (3) implies

$$|\phi'(z)| \leqq \frac{1-|\phi(z)|^2}{1-|z|^2}.$$

Therefore, putting $|z|=r$, $|k|=|z\phi(z)|=u$ and noting that

$$1-4u^2-4u^3-u^4 = (1+u)^2(1-2u-u^2),$$

(4) $$\begin{aligned} Rf'(z) &\geqq \left\{1-2u-u^2+\frac{2u^2}{1-r^2}-\frac{2r^2}{1-r^2}\right\}\frac{(1+u)^2}{|1+k|^4} \\ &= h(u)(1+u)^2|1+k|^{-4}. \end{aligned}$$

By (3),

(5) $$0 \leqq u \leqq r.$$

If $r \leqq 2^{1/2}-1$, $u \leqq r$,

$$\begin{aligned} \tfrac{1}{2}h'(u) &= \frac{1+r^2}{1-r^2}u-1 \\ &\leqq r\frac{1+r^2}{1-r^2}-1 \\ &\leqq (2^{1/2}-1)\frac{1+(2^{1/2}-1)^2}{1-(2^{1/2}-1)^2}-1 = 1-2^{1/2} < 0, \end{aligned}$$

so that

$$h(u) \geqq h(r) = 1 - 2r - r^2 \geqq 0 \qquad (0 < r \leqq 2^{1/2} - 1).$$

Therefore, by (4) and (5),

$$\begin{aligned} \mathbf{R} f'(z) &\geqq (1 - 2r - r^2)(1 + u)^2 \left| 1 + k \right|^{-4} \\ &\geqq (1 - 2r - r^2)(1 + u)^2 (1 + u)^{-4} \\ &\geqq (1 - 2r - r^2)(1 + r)^{-2}. \end{aligned}$$

The function

$$f(z) = z \frac{1 - z}{1 + z} \in S_0$$

has

$$f'(r) = \left\{ 1 - 2r - r^2 \right\} (1 + r)^{-2},$$

which shows that our result is sharp.

3. **Applications.** Theorem 1 gives a new proof of

THEOREM 2 (NOSHIRO, [**5**]). *If $f(z) \in S_0$, then $f(z)$ is univalent in* $|z| < 2^{1/2} - 1$.

This follows by combining the obvious consequence

$$\mathbf{R} f'(z) > 0 \qquad (|z| < 2^{1/2} - 1)$$

with the well-known

WOLFF-NOSHIRO LEMMA. *If $f(z)$ is analytic in $|z| < R$ and $\mathbf{R} f'(z) > 0$ ($|z| < R$), then $f(z)$ is univalent in $|z| < R$.*

THEOREM 3. *If*

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in S_0,$$

then the partial sums

$$s_n(z) = z + a_2 z^2 \cdots + a_n z^n \qquad (n = 2, 3, \cdots)$$

are univalent in $|z| < 1/4$.

PROOF. The function

$$g(z) = f(z)/z = 1 + a_2 z + a_3 z^2 \cdots$$

satisfies (1). It is well known that this implies [**4**, p. 170]

$$|a_n| \leqq 2 \qquad (n = 2, 3, \cdots). \tag{6}$$

Therefore, putting $|z| = r$,

$$\begin{aligned} \mathbf{R}s_n'(z) &= \mathbf{R}f'(z) - \mathbf{R}\sum_{k=n+1}^{\infty} ka_k z^{k-1} \\ &\geqq \mathbf{R}f'(z) - 2\sum_{k=n+1}^{\infty} kr^{k-1}. \end{aligned}$$

Summing the series on the right-hand side and using Theorem 1 we obtain that, for $|z| < 2^{1/2}-1$,

$$\mathbf{R}s_n'(z) \geqq (1-2r-r^2)(1+r)^{-2} - 2[n+1-nr]r^n(1-r)^{-2}.$$

If $r = 1/4$ $(< 2^{1/2}-1)$, then

$$(n+1-nr)r^n \leqq (4-3r)r^3 \qquad (n = 3, 4, 5 \cdots)$$

and so, for $r = 1/4$,

$$\mathbf{R}s_n'(z) \geqq (7/25) - (13/72) > 0.$$

By the minimum modulus principle applied to the harmonic function $\mathbf{R}s_n'(z)$ this implies

$$\mathbf{R}s_n'(z) > 0 \qquad (|z| \leqq 1/4).$$

The result now follows from the Wolff-Noshiro Lemma for $n \geqq 3$. For $n = 2$ we have, by (6),

$$\mathbf{R}s_2'(z) = \mathbf{R}(1 + 2a_2 z) > 1 - 4|z| > 0 \qquad (|z| < 1/4).$$

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References

1. C. Carathéodory, *Theory of functions of a complex variable*. II, Chelsea, New York, 1954.

2. E. Hille, *Analytic function theory*. II, Ginn, Boston, 1962.

3. T. MacGregor, *The radius of convexity for starlike functions of order* 1/2, Proc. Amer. Math. Soc. **14** (1963), 71–76.

4. Z. Nehari, *Conformal mapping*, McGraw-Hill, New York, 1952; p. 170.

5. K. Noshiro, *On the univalency of certain analytic functions*, J. Fac. Sci. Hokkaido Imp. Univ. **2** (1934), 89–101.

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