INNER FACTORS AND BLASCHKE PRODUCTS

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1. Introduction. Let $L_{2,v}$ denote the space of functions on the positive real axis into a separable Hilbert space $V$, such that $\int_0^\infty |f(x)|^2dx < \infty$ and for every $u \in V$, $(f(x), u)$ is measurable. Let $L_{2,v}^\wedge$ denote the space of Fourier transforms

$$\hat{f}(t) = \int_0^\infty f(x)e^{itz} dx,$$

where $f \in L_{2,v}$ and $dx$ denotes $dx/2\pi$. The functions in $L_{2,v}^\wedge$ can be extended to analytic functions in the upper half plane. We denote this space of analytic functions by $H_v$. By the Paley-Wiener theorem, $H_v$ is characterized by the property that $h \in H_v$ if and only if for some constant $M$ and every $y > 0$, $\int_0^\infty (h(x+iy), h(x+iy))dx < M$. $H_v$ is a Hilbert space with inner product $(f, g)_1 = \int_0^\infty (f(x), g(x))dx$.

Let $T_s$ (for fixed $s > 0$) be the left translation operator on $L_{2,v}$, $T_s f(x) = f(x+s)$. The family $\{T_s | s \geq 0\}$ is a semigroup of operators.

Let $r_s$ (for fixed $s > 0$) be the right translation operator on $L_{2,v}$:

$$r_s g(x) = \begin{cases} g(x - s), & \text{for } x - s \geq 0, \\ 0, & \text{for } x - s < 0. \end{cases}$$

The family $\{r_s | s > 0\}$ is a semigroup of isometric operators.

A subspace $l$ of $L_{2,v}$ is said to be left invariant (an $l$-space) if for every $f \in l$, $\{T_s f | s \geq 0\} \subseteq l$.

A subspace $r$ of $L_{2,v}$ is said to be right invariant (an $r$-space) if for every $g \in r$, $\{r_s g | s > 0\} \subseteq r$. It is easily seen that the orthogonal complement of an $r$-space is an $l$-space and vice-versa.

An inner factor is an operator valued function defined and analytic in the upper half plane such that for each $z$, $F(z) : W \rightarrow V$ (where $W$ and $V$ are separable Hilbert spaces), $\|F(z)\| \leq 1$, and for almost all real $z$, $F(z)$ is an isometry. If $V$ is finite dimensional, and $W = V$ then $\det F$ (defined as the determinant of the matrix $(F(z)u_i, u_i)$, where $u_i$ is an orthonormal basis for $V$) is a scalar inner factor.

Let $R$ denote the Fourier transform space of an $r$-space. An $R$-space is characterized as being invariant under multiplication by

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Every (nonzero) closed \( R \)-space can be represented in the form \( R = F^vH_w \), where \( F^v \) is an inner factor and \( F^v(z) : W \to V \).

The inner factor corresponding to an \( R \)-space is unique to within multiplication on the right by a constant unitary operator. By regarding two inner factors as equivalent if they differ on the right by a constant unitary operator, we obtain a one-to-one correspondence between nonzero closed \( R \)-spaces and inner factors. This correspondence carries over to closed \( l \)-spaces (\( l \subset L_2 \)), where \( R \) is the Fourier transform space of \( r (=l^1) \) and will be denoted by a common subscript \( l \sim F_a \) (or lack of subscript, \( l \sim F \)). From the division theory of \([4]\), we have

Theorem B. \( l \subseteq l_b \) if and only if there exists an inner factor \( F_c \) such that \( F_b = F_aF_c \).

A (generalized) exponential is a function \( f \in L_2 \) of the form \( f = p(x)e^{i\lambda x} \) where \( p(x) \) is a polynomial with coefficients in \( V \), and \( \text{Im} \lambda > 0 \). We define the order of the exponential \( f \) as the degree of \( p(x) \); \( \lambda \) is called the exponent belonging to the exponential \( f \).

Definition. An inner factor \( F \) is said to be a Blaschke product if \( l \sim F \) is spanned by the exponentials contained in \( l \).

We show that in the case of scalar inner factors, our definition of a Blaschke product is equivalent to the standard definition.

Let \( \{v_j | j \in J\} \) denote the smallest closed \( l \)-space containing \( \{v_j | j \in J\} \). Let \( l = \{x^ke^{i\lambda x} | j \in J\} \); we note that

\[
[x^ke^{i\lambda x} | j \in J] = [p_{k_j} (x)e^{i\lambda x} | j \in J],
\]

where \( p_{k_j} \) is a polynomial of degree \( k_j \).

Let \( F \) be a Blaschke product according to the standard definition. We will show that \( l \sim F \) has a basis of exponentials. Let \( g \) be an arbitrary element of \( l \), and \( Fh = g \). If \( F \) has a zero of order \( m_j \) at \( z = -\lambda_j \), then it follows from the equation

\[
(Fh)^{k_i}(-\lambda_j) = \int_0^\infty g(x)(ix)^{k_i}e^{-i\lambda_j x} dx,
\]

that \( l_c = \{x^{m_i}e^{i\lambda_j x} | j \in J\} \subseteq l \).

We need only show that \( l \subseteq l_c \). If we substitute \( F_c \) \( \sim l_c \) for \( F \) in equation (1.1), and let \( g \) be an arbitrary element of \( l^1 \), we see that \( F \) has a zero of order \( m_j \) at \( z = -\lambda_j \). It then follows that \( F_c/F \) is an
inner factor. Thus $F_e$ is of the form $F_e = F F_e$, so that by Theorem B, $l \subseteq l_e$.

Conversely, we let $l = \{x^{m_j-1} e^{i \beta_j z} | j \in J\}$, and show that $F (\sim l)$ is a standard Blaschke product. It follows from Equation (1.1) that $F$ has a zero of order $m_j$ at $z = \bar{\lambda}_j$. Let $F_e$ be the standard Blaschke product with zeros of order $m_j$ at $z = \bar{\lambda}_j$. Then $F / F_e$ is an inner factor, so that $F = F_e F_a$. According to Theorem B, $l_e \subseteq l$. From equation (1.1) we see that $l \subseteq l_e$, so that $l_e = l$. This implies that $F_a$ is constant, and completes the proof.

**Definition.** An inner factor $F(z)$ is said to be nonsingular if $\det F(z)$ has no zeros (in the upper half plane).

The following results are well known, but we sketch the proof of Theorem D for the sake of completion:

**Theorem C.** Every scalar inner factor $F$ can be factored in the form $F = F_e F_n$, where $F_e$ is a Blaschke product and $F_n$ is nonsingular. $F_e$ and $F_n$ are uniquely determined to within a constant unitary factor.

**Theorem D.** If $F$ is a scalar Blaschke product, and $F = F_a F_b$ then $F_a$ and $F_b$ are Blaschke products.

**Proof.** We factor $F_a$ and $F_b$ as in Theorem C to obtain $F_a = F_1 F_2$, $F_b = F_3 F_4$ (where $F_1$ and $F_3$ are Blaschke products and $F_2$ and $F_4$ are nonsingular). We must show that $F_2$ and $F_4$ are constant. We have $F = (F_1 F_3)(F_2 F_4)$. From Theorem C we see that $F_2 F_4$ is a constant unit. It then follows from the properties of inner factors that $F_2$ and $F_4$ are constant units.

2. **Results.** We consider the case where $V = W$ and is finite dimensional, and we prove the following two theorems:

**Theorem 1.** Every inner factor $F$ can be factored in the form $F = F_e F_n$ where $F_e$ is a Blaschke product and $F_n$ is nonsingular.

**Theorem 2.** $F$ is a Blaschke product if and only if $\det F$ is a Blaschke product.

We now have a generalization of Theorem D in the

**Corollary.** If $F$ is a Blaschke product and $F = F_a F_b$ where $F_a$ and $F_b$ are inner factors, then $F_a$ and $F_b$ are Blaschke products.

To prove these results we need the following three lemmas; their proofs will be given in §3.

**Lemma 1.** If $\det F$ is constant, then $F$ is a constant unitary operator.
**Lemma 2.** Let $l_\lambda$ be the span of the exponentials (with exponent $\lambda$) which are contained in $l$, and let $F \sim l$. Then $\dim l_\lambda$ equals the order of the zero of $\det F(z)$ at $z = -\overline{\lambda}$.

**Lemma 3.** Let $\beta(z)$ be the Blaschke product factor of $\det F$. Let $l \sim F$, $l_\beta \sim \beta(z)I$, and let $l_\epsilon$ be the span of the exponentials which are contained in $l$. Then $l_\epsilon \subseteq l_\beta$.

**Proof of Theorem 1.** Let $l_\epsilon$ be the span of the exponentials contained in $l$ ($\sim F$), and let $F_e \sim l_\epsilon$. According to Theorem B, there exists an inner factor $F_n$ such that $F = F_e F_n$. Taking determinants of both sides, we have $\det F = \det F_e \det F_n$. Let $d$, $d_\epsilon$, and $d_n$ denote the order of the zero at $z = -\overline{\lambda}$ of $\det F$, $\det F_e$, and $\det F_n$ respectively, so that $d = d_\epsilon + d_n$. We must show that $d_n = 0$. Since $l$ and $l_\epsilon$ have the same number of linearly independent exponentials with exponent $\lambda$, it follows from Lemma 2 that $d = d_\epsilon$.

**Proof of Theorem 2.** Assume that $\det F$ is a Blaschke product. We factor $F$ as in Theorem 1: $F = F_e F_n$. We must show that $F_n$ is constant. Taking determinants, we have $\det F = \det F_e \det F_n$. By virtue of Theorem D, $\det F_n$ is a Blaschke product. But since according to Theorem 1, $\det F_n$ has no zeros, it follows that $\det F_n$ is constant. Then by virtue of Lemma 1, $F_n$ is constant.

Conversely, assume that $F$ is a Blaschke product. Let $\beta(z)$ be the Blaschke product factor of $\det F$. Let $l \sim F$ and $l_\beta \sim \beta(z)I$. According to Lemma 3, $l \subseteq l_\beta$. Then by Theorem B, there exists an inner factor $F_a$ such that $\beta(z)I = FF_a$. Taking determinants, we have $\beta^n(z) = \det F \det F_a$ (where $n = \dim V$). By virtue of Theorem D, $\det F$ is a Blaschke product.

3. Proofs of lemmas.

**Proof of Lemma 1.** Since $\det F(z) (= c)$ is an inner factor, we have $\det (F^*(z)F(z)) = c c = 1$. Let $\{a_j(z) \mid j = 1, \ldots, n\}$ be the eigenvalues of $F^*(z)F(z)$. Since $\|F^*(z)F(z)\| \leq 1$ (for $\Im z > 0$), and $F^*(z)F(z)$ is nonnegative, we have $0 \leq a_j(z) \leq 1$ ($j = 1, \ldots, n$). It then follows from the equation $\det (F^*(z)F(z)) = \prod_{j=1}^n a_j(z) = 1$ that all the eigenvalues of $F^*(z)F(z)$ are equal to one. Since $F^*(z)F(z)$ is symmetric, we thus have $F^*(z)F(z) = I$. Similarly $F(z)F^*(z) = I$, so that $F^*(z) = F^{-1}(z)$. Since $F(z)$ is analytic and $F^{-1}(z)$ is continuous, $F^{-1}(z)$ is analytic. But if $F(z)$ and $F^*(z)$ ($= F^{-1}(z)$) are both analytic, it follows that $F(z)$ is constant.

The proof of Lemma 2 is based on four sublemmas:

**Lemma 3.1.** Let $l = [ue^{i\alpha z}]$ where $u$ is a given vector in $V$ and $\Im > 0$. Then $F (\sim l)$ is (up to a constant unitary operator on the right) of the form...
(3.1) \[ F = P + b(z)Q, \]
(3.2) \[ b(z) = \frac{z + \bar{\lambda}}{z + \lambda}, \]

Q is the orthogonal projection onto the one dimensional space spanned by u, and \( P = I - Q. \)

**Proof.** Let \( l_p \sim P + b(z)Q. \) Then

\[ \langle (P + b(z)Q)h_v, u \rangle = \langle h_v, P* u \rangle + b(z) \langle h_v, Q* u \rangle = b(z) \langle h_v, u \rangle. \]

Since \( b(-\bar{\lambda}) = 0, \) it follows from the equation

\[ 0 = \langle (P + b(-\bar{\lambda})Q)h_v(-\bar{\lambda}), u \rangle = \int_0^\infty (a_v, u)e^{-\bar{\lambda}x} \, dx \]

(where \( a_v \in l^1 \)) that \( l = [ue^{\bar{\lambda}x}] \subseteq l_p. \) From Theorem B we obtain \( P + b(z)Q = FF_a (where F \sim l). \) Taking determinants of both sides of the factorization equation we have \( b(z) = \det F \cdot \det F_a. \) It follows from Theorem D that either \( \det F \) or \( \det F_a \) is constant. If \( \det F \) were constant, then by Lemma 1, \( F \) would be a constant unitary operator. Clearly, this would imply that \( l = \{0\}. \) Thus by contradiction we see that \( \det F_a, \) and therefore \( F_a, \) must be constant, so that \( l = l_p. \)

**Definition.** An inner factor of the form (3.1) is called a prime inner factor (at \( \lambda \)).

**Lemma 3.2.** Let \( F \) be an inner factor, and let \( \det F \) have a zero of order at least \( m \) at \( z = -\bar{\lambda}. \) Then \( F \) can be factored in the form \( F = (\prod_{j=1}^m F_j)F_a \) for some inner factor \( F_a, \) where \( \{ F_j | j = 1, \ldots, m \} \) are prime inner factors (at \( \lambda \)).

**Proof.** We use induction. Take \( m = 1. \) Since \( \det F(-\bar{\lambda}) = 0, \) there exists a vector \( u (u \neq 0) \) such that \( F^*(-\bar{\lambda})u = 0. \) Let \( l \sim F. \) It follows from the equation

\[ 0 = \langle h_v(-\bar{\lambda}), F^*(-\bar{\lambda})u \rangle = \langle F(-\bar{\lambda})h_v(-\bar{\lambda}), u \rangle = \int_0^\infty (a_v, u)e^{-\bar{\lambda}x} \, dx \]

(where \( a_v \in l^1 \)) that \( ue^{\bar{\lambda}x} \subseteq l. \) Let \( l_1 = [ue^{\bar{\lambda}x}], \) so that \( l_1 \subseteq l. \) Then by Theorem B, we have \( F = F_1F_a, \) where \( F_1 \sim l_1. \) By virtue of Lemma 3.1, \( F_1 \) is a prime inner factor (at \( \lambda \)).

We now assume that the lemma is true for \( m = k, \) and consider the case \( m = k + 1. \) By our assumption we have \( F = (\prod_{j=1}^k F_j)F_a. \) Taking determinants, we have \( \det F = b^k(z) \cdot \det F_a. \) Since the order of the zero of \( b^k(z) \) at \( z = -\bar{\lambda} \) is \( k, \) we must have \( \det F_a(-\bar{\lambda}) = 0. \) Then,
as shown above, \( F_a = F_{k+1}F_b \) for some inner factor \( F_b \), where \( F_{k+1} \) is a prime inner factor (at \( \lambda \)).

**Lemma 3.3.** The space \( l = \{ux^{m-1}e^{i\lambda x} \mid u \in V \} \) corresponds to the inner factor \( F_b = b^m(z)I \).

Let \( l_1 = \{ux^{m-1}e^{i\lambda x} \mid u \in V \} \). It follows from the equation

\[
(F(z)h_v, u)^{(k)} = \int_0^\infty (a_v, u)(ix)^k e^{i\lambda x} \, dx,
\]

for \( a_v \in l_1^* \), and \( F = F_1 (\sim l_1) \), that \( F_1 \) has a zero of order at least \( m \) at \( z = -\lambda \). That is, \( F_1^{(k)}(-\lambda) = 0 \), \( k = 0, 1, \ldots, m - 1 \). Then we can factor \( F_1 \) in the form \( F_1 = c b^m(z)I \cdot F_c \), where \( F_c \) is an inner factor. We need only show that \( F_c \) is constant. Let \( h_b \sim F_b = b^m(z)I \). It is easy to see from equation (3.3), for \( \gamma_v = l_1^* \), and \( F = F_b \), that \( \gamma_v \) is constant. Let \( h_b \sim F_b = b^m(z)I \). It follows that \( F_c \) is a constant unitary operator.

**Lemma 3.4.** \( l (\sim F) \) consists of exponentials with exponent \( \lambda \) and is finite dimensional if and only if \( \det F = cb^m(z) \). Also, if \( \det F = cb^m(z) \), then \( \dim l = m \).

Let \( l \) consist of exponentials with exponent \( \lambda \), and let \( q - 1 \) be the order of the highest order exponential contained in \( l \). Let \( l_b = \{ux^{q-1}e^{i\lambda x} \mid u \in V \} \). Then \( l \subseteq l_b \). According to Lemma 3.3, \( b^q(z)I \sim l_b \).

By virtue of Theorem B, we have \( b^q(z)I = FF_a \) (where \( F \sim l \)). Taking determinants, we obtain \( b^{qn}(z) = \det F \cdot \det F_a \) (where \( n = \dim V \)). It follows from Theorem D that \( \det F \) is some power of \( cb(z) \).

Conversely, let \( \det F = cb^m(z) \). According to Lemma 3.2 we can factor \( F \) in the form

\[
F = \left( \prod_{j=1}^m F_j \right) F_a,
\]

where \( F_j \) is the prime inner factor \( P_j + b(z)Q_j \). By taking determinants of both sides of equation (3.4), we see that \( \det F_a = c \). Then by virtue of Lemma 1, \( F_a \) is a constant unitary operator. Let \( E_j = Q_j + b(z)P_j \) \( (j = 1, \ldots, m) \). Clearly each \( E_j \) is an inner factor. Since

\[
\left( \prod_{j=1}^m F_j \right) F_a F_a^* \left( \prod_{j=0}^{m-1} E_{m-j} \right) = b^m(z)I,
\]

it follows from Theorem B that \( l \subseteq l_b \). It then follows from Lemma 3.3 that \( l \) consists of exponentials with exponent \( \lambda \).
Again, we assume that \( \det F = cb^m(z) \). We will show that \( \dim l = m \). As shown in the previous paragraph,
\[
I \left( \sim \prod_{j=1}^{m} F_j \right) \subseteq l_b \left( \sim b^m(z)I \right).
\]

According to Theorem B, we have \( b^m(z)I = \left( \prod_{j=1}^{m} F_j \right) F_a \). Taking determinants, we have \( b^m(z) = b^m(z) \cdot \det F_a(z) \), so that \( \det F_a(z) = b^m(z) \cdot \det F_a(z) \). By virtue of Lemma 3.2, \( F_a \) can be factored in the form \( F_a = \prod_{j=m+1}^{mn} F_j \). Let \( E_k = \prod_{j=1}^{k} F_j \) \((k \leq mn)\), and let \( l_n = l \) \((l_m = l, \text{and } l_{mn} = l_b)\). From Theorem B, we have \( l_{k-1} \subseteq l_k \). Since no \( F_j \) is constant, we have \( l_{k-1} \subseteq l_k \). By use of induction, we see that \( \dim l_k \geq k \), and also that \( \dim l_{p+k} \geq \dim l_p + k \), \((k + p \leq mn)\). If \( \dim l_k > k \) for some fixed \( k \), then \( \dim l_{mn} \geq \dim l_k + mn - k > mn \). But it follows from Lemma 3.3 that \( \dim l_b = mn \). Thus, by contradiction, \( \dim l_k = k \), \( k = 1, \ldots, mn \).

**Proof of Lemma 2.** Let \( l \) contain (exactly) \( m \) linearly independent exponentials with exponent \( \lambda \), and let \( l_\lambda \) be the span of these exponentials. From Theorem B, we have \( F = F_\lambda F_a \). Taking determinants of both sides, we have \( \det F = \det F_\lambda \cdot \det F_a \). According to Lemma 3.4, \( \det F_\lambda = cb^m(z) \), so that \( \det F \) has a zero of order at least \( m \) at \( z = -\lambda \).

Conversely, we assume that \( \det F \) has a zero of order \( m \) at \( z = -\lambda \). According to Lemma 3.2, \( F \) can be factored in the form \( F = \left( \prod_{j=1}^{m} F_j \right) F_a \) for prime \( F_j \)'s. Let \( l_m = \prod_{j=1}^{m} F_j \). Since \( \det \prod_{j=1}^{m} F_j = b^m(z) \), it follows from Lemma 3.4 that \( l_m \) contains \( m \) linearly independent exponentials with exponent \( \lambda \). Then by Theorem B, we have \( l_m \subseteq l \), which completes the proof.

**Proof of Lemma 3.** Let \( l \) contain an exponential \( p(x)e^{ikx} \) of order \( m \). Since \( (T_\ast - e^{ikx})p(x)e^{ikx} \((\in l)\) is an exponential of order \( m - 1 \), we see that \( l \) contains at least \( m + 1 \) linearly independent exponentials with exponent \( \lambda \). By virtue of Lemma 2, \( \beta(z) \) has a zero of order at least \( m + 1 \), so that
\[
(\beta(-\lambda)I \cdot h_\ast(-\lambda),u)^{(k)} = 0, \quad \text{for } k = 0, \ldots, m.
\]

It then follows from equation (3.3) that all exponentials with exponent \( \lambda \) of order \( m \) or less are contained in \( l_\beta \).

**Bibliography**


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