ON THE SQUARES OF ORIENTED MANIFOLDS

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1. **Introduction.** The object of this paper is to give another proof of the Milnor conjecture [2]:

THEOREM 1. *The square of an oriented manifold is unoriented cobordant to a Spin manifold.*

This result was proved by Anderson [1]. The proof given here is patterned directly on the method used by Wall [4] in the determination of the oriented cobordism ring. This method provides additional information, and the principal result will be:

THEOREM 2. *Let S' denote the subset of $H^*(BO, Z_2)$ consisting of all classes w_{2i+1}. Let S be either $S'\cup\{w_2\}$ or $S'\cup\{w_2^2\}$. Let M be a manifold such that every Stiefel-Whitney number of M divisible by a class of S is zero. Then:*

(a) *For $S=S'\cup\{w_2^2\}$, M is unoriented cobordant to a complex manifold M' with c_1^2 zero.*

(b) *For $S=S'\cup\{w_2\}$, M is unoriented cobordant to the sum of an* SU *manifold and a polynomial in the quaternionic projective spaces* QP$(2n)$.

Theorem 1 is a direct consequence of the case $S=S'\cup\{w_2\}$, and one has the improved result:

COROLLARY. *If M is an oriented manifold with dimension not divisible by* 4, *then $M\times M$ is unoriented cobordant to an* SU *manifold.*

2. **Proofs of the results.** First consider the case $S=S'\cup\{w_2^2\}$ and suppose M has all Stiefel-Whitney numbers divisible by elements of S zero.

By Milnor [2], M is cobordant to a complex manifold N, since all numbers of M divisible by elements of S' are zero. Then let $M'\subset N \times \mathrm{CP}(1)$ be a submanifold dual to $c_1(N)+\alpha$, $\alpha\in H^2(\mathrm{CP}(1), Z)$ being the usual generator. The total Chern class of M' is the restriction to M' of

$$\frac{c(N)\cdot(1+\alpha)^2}{1+\alpha+c_1(N)} = 1+\alpha+(c_2(N)+\alpha c_1(N))+\cdots,$$

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so $c_1^2(M')$ is the restriction to M' of α^2, hence is zero. Reducing mod 2,

$$w(M') = \frac{w(N)}{1 + \alpha + w_2(N)},$$

and M' is dual to $\alpha + w_2(N)$. Then for any $\omega = (i_1, \cdots, i_r)$, $\sum i_s = \dim M$, $w_\omega = w_{i_1} \cdots w_{i_r}$, one has

$$w_\omega(M') = w_\omega(N) + (\alpha + w_2(N))v_\omega,$$

where v_ω is a polynomial in α and the $w_i(N)$. Multiplying by $(\alpha + w_2(N))$ and evaluating on $N \times \mathrm{CP}(1)$ gives

$$\begin{aligned} w_\omega[M'] &= \{w_\omega(N)\cdot(\alpha + w_2(N)) + (\alpha + w_2(N))^2 v_\omega\}[N \times \mathrm{CP}(1)], \\ &= \{w_\omega(N)\cdot\alpha + w_2(N)^2 v_\omega\}[N \times \mathrm{CP}(1)], \end{aligned}$$

and the second term is zero since every number of N divisible by w_2^2 is zero. Then $w_\omega[M'] = w_\omega[N]$ for all ω, and by Thom [**3**], M' and N are cobordant since they have the same Stiefel-Whitney numbers. This proves part (a) of the theorem.

Note. Using the same process with $\mathrm{RP}(1)$ and its class α gives a direct proof of Wall's result [**4**] that if M has all numbers divisible by w_1^2 zero, then M is cobordant to M' with $w_1^2 = 0$.

Let $\mathcal{W}_*$ denote Wall's ring of cobordism classes for which numbers divisible by w_1^2 are zero. Since

$$w_{i_1} \cdots w_{i_r}[N \times N] = \begin{cases} 0 & \text{if any } i_s \text{ is odd,} \\ w_{j_1} \cdots w_{j_r}[N] & \text{if } i_s = 2j_s \text{ for all } s, \end{cases}$$

one has $\mathcal{W}_*^2 = \{\alpha^2 \mid \alpha \in \mathcal{W}_*\}$ is the set of cobordism classes with numbers divisible by elements of $S' \cup \{w_2^2\}$ zero.

Let $\partial_1 \colon \mathcal{W}_* \to \mathcal{W}_*$ be the derivation defined by sending the cobordism class of M to the class of a submanifold dual to w_1. If $\partial_2 \colon \mathcal{W}_*^2 \to \mathcal{W}_*^2 \colon \alpha^2 \to (\partial_1\alpha)^2$, then ∂_2 is a derivation and sends the cobordism class of M to the class of a manifold N such that $w_\omega[N] = w_2 \cdot w_\omega[M]$ for all ω (by the formula for $w_\omega[A \times A]$).

Now if $\alpha^2 \in \mathcal{W}_*^2$, let M be a complex manifold with $c_1^2 = 0$ belonging to α^2. Let N be a submanifold dual to c_1 in M. Then $w_\omega[N] = w_2 \cdot w_\omega[M]$ for all ω, so N belongs to $(\partial_1\alpha)^2 = \partial_2(\alpha^2)$, and N is an SU manifold. Thus $\operatorname{Im} \partial_2$ consists entirely of cobordism classes of SU manifolds.

By Wall [**4**], $\ker \partial_1 / \operatorname{Im} \partial_1$ is the Z_2 polynomial algebra on the images of the cobordism classes of the complex projective spaces $\mathrm{CP}(2n)$, so $\ker \partial_2 / \operatorname{Im} \partial_2$ is the Z_2 polynomial algebra on the images of the cobordism classes of the $CP(2n)^2$ (which is the same as that of $QP(2n)$).

Now let M be any manifold all of whose Stiefel-Whitney numbers divisible by elements of $S' \cup \{w_2\}$ are zero. Then the cobordism class α of M belongs to ker ∂_2. Thus there is a polynomial P in the $QP(2n)$ with cobordism class γ for which $\alpha - \gamma \in \operatorname{Im} \partial_2$. Hence there is an SU manifold N such that M is cobordant to $N+P$, proving part (b) of Theorem 2.

Note. $\ker \partial_2 = \{\alpha^2 \mid \alpha \in \ker \partial_1\}$ is precisely the cobordism classes of squares of oriented manifolds.

References

1. P. G. Anderson, *Cobordism classes of squares of orientable manifolds*, Bull. Amer. Math. Soc. **70** (1964), 818–819.

2. J. Milnor, *On the Stiefel-Whitney numbers of complex manifolds and of spin manifolds*, Topology **3** (1965), 223–230.

3. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954), 17–86.

4. C. T. C. Wall, *Determination of the cobordism ring*, Ann. of Math. (2) **72** (1960), 292–311.

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