ON THE BOUNDARY BEHAVIOR OF BLASCHKE PRODUCTS IN THE UNIT DISK

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1. **Introduction.** Let A be any sequence of points in the unit disk D: $\{|z|<1\}$ such that $\sum_A (1-|a|)<\infty$. The Blaschke product with respect to A, given by $B(z;A)=\prod_A (|a|/a)\,(a-z)/(1-\bar{a}z)$, defines an analytic function in D such that $|B(z;A)|<1$ for z in D. F. Riesz [**7**, p. 94] showed that for almost all points $e^{i\theta}$ of C: $\{|z|=1\}$ the radial limit $B(e^{i\theta})=\lim_{r\to 1} B(re^{i\theta};A)$ exists and is of modulus one. For later reference we state the following result of O. Frostman [**4**, p. 170].

THEOREM 1. *A necessary and sufficient condition that $B(z;A)$ and all it subproducts have radial limits of modulus one at $e^{i\theta}$ is that*

$$\sum_A [(1-|a|)/|e^{i\theta}-a|]<\infty. \tag{1}$$

It is our purpose to consider the boundary behavior of Blaschke products possessing radial limits of modulus one at every point of C.

2. Because of condition (1), it is only at accumulation points of A that a Blaschke product $B(z;A)$ can possibly fail to have a radial limit of modulus one. The following theorem gives the restrictions to be imposed on A', the derived set of A, in order that $B(z;A)$ have radial limits of modulus one at every point of C.

THEOREM 2. *Let E be a set on C. A necessary and sufficient condition that there exist a Blaschke product $B(z;A)$ for which $B(e^{i\theta})$ is defined and of modulus one at every point of C and such that $A'=E$ is that E be closed and nowhere dense on C.*

Let E be a closed and nowhere dense set on C. We shall construct a sequence A in D with $A'=E$ for which $\sum_A (1-|a|)<\infty$ and for which (1) is satisfied at every point of C. Then, by Theorem 1, the corresponding Blaschke product $B(z;A)$ will have radial limits of modulus one at every point of C.

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The complement of E with respect to C is a countable union of disjoint open arcs on C. Let F be the set of end-points of these open arcs, where we write $F=\{c_m, d_m | m=1, 2, 3, \cdots,\}$, with $|c_m|=|d_m| =1$, $\xi_m=\arg c_m<\arg d_m=\delta_m$ for each m. It is clear that F is a countable dense subset of E.

For each m, let $l_m=\delta_m-\xi_m$, and let a number t, $0<t<1$, be chosen. Define k_m for each m to be the minimum of t and $l_m/2\pi$. For each m and $n=1, 2, 3, \cdots$, we define the sequences $\{g_{mn}\}$ and $\{h_{mn}\}$ by $g_{mn}=\xi_m+(k_m)^n/2$, $h_{mn}=\delta_m-(k_m)^n/2$. For each fixed m, $\{g_{mn}\}$ is nonincreasing in n, $\{h_{mn}\}$ is nondecreasing in n, while $\lim_{n\to\infty} g_{mn}=\xi_m$ and $\lim_{n\to\infty} h_{mn}=\delta_m$. For each m and n we let $r_{mn}=1-(k_m)^n/2^{m+n}$, and we let $A=\{a_{mn}, b_{mn} | m, n=1, 2, 3, \cdots,\}$, where $a_{mn}=r_{mn}\exp(ig_{mn})$ and $b_{mn}=r_{mn}\exp(ih_{mn})$.

It may be seen that $A'=E$ and $\sum_A (1-|a|)<\infty$. For any point $e^{i\theta}$ of $C-E$, clearly (1) is satisfied. If $e^{i\theta}$ is a point of E, we see that $\theta\neq\arg a$ for each a in A. If we denote by $|\alpha-\beta|$ the length along C of the shorter arc from $e^{i\alpha}$ to $e^{i\beta}$, then $|\theta-\arg a|\neq 0$ for $e^{i\theta}$ in E and a in A. By a lemma of G. T. Cargo [**2**, p. 10], to show (1) is satisfied for $e^{i\theta}$ in E, it suffices to show that

$$\sum_A [(1-|a|)/|\theta-\arg a|]<\infty. \tag{2}$$

However, for $e^{i\theta}$ in E we have $|\theta-\arg a_{mn}|\geqq(k_m)^n/2$, $|\theta-\arg b_{mn}| \geqq(k_m)^n/2$ for $m=1, 2, 3, \cdots$, $n=1, 2, 3, \cdots$, so that (2) will be satisfied for $e^{i\theta}$ in E.

Let $B(z; A)$ be a Blaschke product with radial limits of modulus one at every point of C, and let $E=A'$.

Of course E is closed, but suppose E is not nowhere dense on C. Then there is some arc I on C in which E is dense. Write $I=\{e^{i\theta} | \alpha \leqq\theta\leqq\beta\}$, and let γ, $0<\gamma<\pi$, be arbitrarily chosen. We denote by S_θ the Stolz angle in D at $e^{i\theta}$ with vertex angle γ symmetric about the radius to $e^{i\theta}$. In the region $\{re^{i\theta} | 0<r<1, \alpha<\theta<\beta\}$ we can select a point a_1 of A. If $\arg a_1=\phi_1$, we may choose α_{11}, α_{12} such that $\alpha\leqq\alpha_{11} <\phi_1<\alpha_{12}\leqq\beta$ and such that a_1 is in S_θ for $\alpha_{11}<\theta<\alpha_{12}$. Now choose β_{11}, β_{12} such that $\alpha_{11}<\beta_{11}<\phi_1<\beta_{12}<\alpha_{12}$, and let $J_1=\{e^{i\theta} | \beta_{11}\leqq\theta\leqq\beta_{12}\}$. Clearly a_1 is in S_θ for every $e^{i\theta}$ in J_1.

In the region $\{re^{i\theta} |\ |a_1|<r<1, \beta_{11}<\theta<\beta_{12}\}$ we can select a point a_2 of A. If $\arg a_2=\phi_2$, we choose α_{21}, α_{22} such that $\beta_{11}\leqq\alpha_{21}<\phi_2<\alpha_{22}\leqq\beta_{12}$ and such that a_2 is in S_θ for $\alpha_{21}<\theta<\alpha_{22}$. Now we choose β_{21}, β_{22} such that $\alpha_{21}<\beta_{21}<\phi_2<\beta_{22}<\alpha_{22}$, and we let $J_2=\{e^{i\theta} | \beta_{21}\leqq\theta\leqq\beta_{22}\}$. We see that $J_2\subset J_1$, while a_1 and a_2 are both in S_θ for all $e^{i\theta}$ in J_2.

Continuing in this fashion, we construct a sequence $\{J_j\}$ of closed arcs on C such that $J_1 \supset J_2 \supset \cdots \supset J_j \supset \cdots$, and we select a sequence $\{a_j\}$ of points in A with $|a_1| < |a_2| < \cdots < |a_j| < \cdots < 1$ and such that for each value of j and for every $e^{i\theta}$ in J_j, a_k is in S_θ, $k = 1, 2, \cdots, j$.

Now $\bigcap J_j$, where the intersection is taken over all values of j, is not empty, and we can find a point $e^{i\phi}$ of $\bigcap J_j$ which is an accumulation point of $\{a_j\}$. Also, for each value of j, a_j is in S_ϕ.

We connect the points of $\{a_j\}$ in order of increasing index by a polygonal path $P(z)$ to $e^{i\phi}$ lying in S_ϕ. The limit of $B(z; A)$ as z approaches $e^{i\phi}$ along $P(z)$, if it exists, cannot be of modulus one, for $B(a_j; A) = 0$ for $j = 1, 2, 3, \cdots$. An application of a theorem of E. Lindelöf [**5**] shows that $B(z; A)$ cannot then have a radial limit of modulus one at $e^{i\phi}$. From this contradiction we conclude that E must be nowhere dense on C.

The proof that E is necessarily nowhere dense on C, while cumbersome, uses only elementary techniques. By appealing to cluster set theory, a far more elegant proof is possible. The author is indebted to Professor K. Noshiro for the following alternative proof of the fact that E is nowhere dense on C.

Each point $e^{i\theta}$ of E is an accumulation point of the zeros of $B(z; A)$ and thus is an essential singularity of $B(z; A)$. By a theorem of W. Seidel [**9**, p. 211], the interior cluster set of $B(z; A)$ at $e^{i\theta}$, $C(B, e^{i\theta})$, is the closed unit disk. However, by hypothesis $B(z; A)$ possesses a radial limit at each point of C, so that radial cluster set for $B(z; A)$ at $e^{i\theta}$, $C_r(B, e^{i\theta})$, is a single point.

Hence at each point $e^{i\theta}$ of E we have $C(B, e^{i\theta}) \neq C_r(B, e^{i\theta})$. By a theorem of E. F. Collingwood [**3**, p. 5], E must be a set of category I on C. Since E is closed, E is necessarily nowhere dense on C.

THEOREM 3. *Let $B(z; A)$ be a Blaschke product with $B(e^{i\theta})$ defined and of modulus one at every point of C. Then, as a function of θ, $B(e^{i\theta})$ is discontinuous at $\theta = \theta_0$ if, and only if, $e^{i\theta_0}$ is in A'.*

If $e^{i\theta_0}$ is not a point of A', then a theorem of C. Tanaka [**10**, p. 410] states that $B(z; A)$ is analytic throughout a neighborhood of $e^{i\theta_0}$. Throughout this neighborhood, $B(e^{i\theta}) = \lim_{r \to 1} B(re^{i\theta}; A) = B(e^{i\theta}; A)$, so that $B(e^{i\theta})$ is evidently continuous at $\theta = \theta_0$.

If $e^{i\theta_0}$ is a point of A', then $e^{i\theta_0}$ is a singularity of $B(z; A)$. Consequently, as was proved by W. Seidel [**9**, p. 208], in each arc on C containing $e^{i\theta_0}$, $B(e^{i\theta})$ assumes every value of modulus one infinitely often. Then $B(e^{i\theta})$ is discontinuous at $\theta = \theta_0$.

From Theorems 2 and 3 follows immediately a corollary which is a special case of a theorem of A. J. Lohwater and G. Piranian [**6**, p. 5].

COROLLARY. *Let $B(z; A)$ be any Blaschke product for which $B(e^{i\theta})$ is defined and of modulus one at every point of C. In order that a set E on C be exactly the set of points where the radial limit function $B(e^{i\theta})$ is discontinuous, it is necessary and sufficient that E be closed and nowhere dense on C.*

3. For a Blaschke product $B(z; A)$ in D, the radial variation of $B(z; A)$ at a point $e^{i\theta}$ of C is defined to be $V(B;\theta)=\int_0^1 |B'(re^{i\theta};A)|\,dr$. The quantity $V(B;\theta)$ is the length of the image under $B(z; A)$ of the radius to $e^{i\theta}$. G. T. Cargo [1, p. 425] proved that (1) is a necessary and sufficient condition for the radial variations of $B(z; A)$ and all its subproducts at $e^{i\theta}$ to be uniformly bounded.

THEOREM 4. *Let $B(z; A)$ be a Blaschke product for which* (1) *holds at every point of C. Then, as a function of θ, $V(B;\theta)$ is a discontinuous at $\theta=\theta_0$ if, and only if, $e^{i\theta_0}$ is in A'.*

Suppose $e^{i\theta_0}$ is not in A'. Then there is an open neighborhood N of $e^{i\theta_0}$ throughout which $B(z; A)$ is analytic, [10, p. 410], and $B(z; A)$ is analytic in $D\cup N$. Consequently, $B'(z; A)$ is defined and continuous throughout $D\cup N$.

If d denotes the distance from $e^{i\theta_0}$ to the closest boundary point of N, let R be the closed region $\{re^{i\theta}\,|\,0\leqq r\leqq 1,\ |e^{i\theta}-e^{i\theta_0}|\leqq d/2\}$. Now $R\subset D\cup N$, and $B'(z; A)$ is uniformly continuous on R. Given any $\epsilon>0$ there exists $\delta>0$ such that for all r, $0\leqq r\leqq 1$, $|B'(re^{i\theta};A)-B'(re^{i\theta_0};A)|<\epsilon$ when $|\theta-\theta_0|<\delta$.

Then for all θ, $|\theta-\theta_0|<\delta$, we have

$$|V(B;\theta)-V(B;\theta_0)| = \left|\int_0^1 |B'(re^{i\theta};A)|\,dr-\int_0^1 |B'(re^{i\theta_0};A)|\,dr\right|$$

$$\leqq \int_0^1 |B'(re^{i\theta};A)-B'(re^{i\theta_0};A)|\,dr<\epsilon,$$

so that $V(B;\theta)$ is continuous at $\theta=\theta_0$.

Suppose $e^{i\theta_0}$ is in A' while $V(B;\theta)$ is continuous at $\theta=\theta_0$. We may select a subsequence $\{r_k\exp(i\phi_k)\,|\,k=1,2,3,\cdots\}$ from A such that $r_k\leqq r_{k+1}$ and $|\phi_k-\theta_0|\geqq|\phi_{k+1}-\theta_0|$ for $k=1, 2, 3, \cdots$, $\lim_{k\to\infty} r_k\exp(i\phi_k)=e^{i\theta_0}$, and $\lim_{k\to\infty} V(B;\phi_k)=V(B;\theta_0)$.

Let s, $0<s<1$, be arbitrarily chosen, and let $k(s)$ be a positive integer such that $r_k\geqq s$ when $k\geqq k(s)$. Since for each value of k, $B[r_k\exp(i\phi_k);A]=0$ while $|B[\exp(i\phi_k)]|=1$ by Theorem 1, we see that for $k\geqq k(s)$, $\int_s^1 |B'[r\exp(i\phi_k);A]|\,dr\geqq 1$.

Thus for $k\geqq k(s)$ we have $V(B;\phi_k)=\int_0^1 |B'[r\exp(i\phi_k);A]|\,dr \geqq\int_0^s |B'[r\exp(i\phi_k);A]|\,dr+1$. Now $B'(z; A)$ is uniformly continuous on the compact subset $\{|z| \leqq s\}$ of D, while for any r, $0 \leqq r \leqq s$, we have $\lim_{k\to\infty} |B'[r \exp(i\phi_k); A]| = |B'(re^{i\theta_0}; A)|$. Consequently, $\lim_{k\to\infty} \int_0^s |B'[r \exp(i\phi_k); A]| dr = \int_0^s \{\lim_{k\to\infty} |B'[r \exp(i\phi_k); A]|\} dr = \int_0^s |B'(re^{i\theta_0}; A)| dr$, and $V(B; \theta_0) = \lim_{k\to\infty} V(B; \phi_k) \geqq \int_0^s |B'(re^{i\theta_0}; A)| dr + 1$, where $0 < s < 1$.

Since (1) holds for $e^{i\theta_0}$, $V(B; \theta_0) < \infty$, and $\lim_{s\to 1} \int_0^s |B'(re^{i\theta_0}; A)| dr = V(B; \theta_0)$, so that $V(B; \theta_0) \geqq V(B; \theta_0) + 1$. We conclude that if $e^{i\theta_0}$ is in A', then $V(B; \theta)$ is discontinuous at $\theta = \theta_0$.

We remark here that since $V(B; \theta)$ is a lower semicontinuous function of θ on $[0, 2\pi]$ (cf. [**8**, p. 235], Theorem 3 implies that $V(B; \theta)$ cannot have a relative maximum at $\theta = \theta_0$ if $e^{i\theta_0}$ is in A'.

4. It is known, [**4**, p. 177], that if a Blaschke product $B(z; A)$ satisfies

$$(3) \qquad \sum_A [(1 - |a|)/|e^{i\theta} - a|^2] < \infty$$

at a point $e^{i\theta}$ of C, then the derivative of $B(z; A)$ has a finite radial limit at $e^{i\theta}$ given by

$$B'(e^{i\theta}) = \lim_{r\to 1} B'(re^{i\theta}; A) = B(e^{i\theta})e^{-i\theta} \sum_A [(1 - |a|^2)/|e^{i\theta} - a|^2],$$

where $B(e^{i\theta}) = \lim_{r\to 1} B(re^{i\theta}; A)$.

At any point $e^{i\theta}$ of C where (3) holds, (1) also holds, and a slight modification of the proof of Theorem 2 justifies

THEOREM 5. *A necessary and sufficient condition that a set E on C be the set of accumulation points of the zeros of a Blaschke product $B(z; A)$ whose derivative has a finite radial limit at every point of C is that E be closed and nowhere dense on C.*

THEOREM 6. *Let $B(z; A)$ be a Blaschke product for which* (3) *holds at every point of C. As a function of θ, $B'(e^{i\theta})$ is discontinuous at $\theta = \theta_0$ if, and only if, $e^{i\theta_0}$ is in A'.*

Let $M(\theta) = \sum_A [(1 - |a|^2)/|e^{i\theta} - a|^2]$ for each θ; we note that for each value of θ each summand of $M(\theta)$ is a continuous function of θ.

Suppose $e^{i\theta_0}$ is not a point of A'. Then for some $\delta > 0$, $|e^{i\theta_0} - a| \geqq \delta$ for all points a of A, and for each point $e^{i\theta}$ of $C \cap \{z \mid |e^{i\theta_0} - z| \leqq \delta/2\}$, we have $|e^{i\theta} - a| \geqq \delta/2$ and $M(\theta) \leqq (8/\delta^2) \sum_A (1 - |a|)$. Thus $M(\theta)$ converges uniformly to a continuous function of θ in $\{\theta \mid |e^{i\theta} - e^{i\theta_0}| \leqq \delta/2\}$.

By Theorem 2, $B(e^{i\theta})$ is continuous at $\theta = \theta_0$, so $B'(e^{i\theta}) = B(e^{i\theta}) \cdot e^{-i\theta}$, $M(\theta)$ is continuous at $\theta = \theta_0$.

Suppose now that $e^{i\theta_0}$ is in A' and $B'(e^{i\theta})$ is continuous at $\theta=\theta_0$. We see that $M(\theta)$ is real-valued, and $M(\theta) \geqq (1/4)\sum_A (1-|a|)$. Further, since (3) holds, $|B(e^{i\theta})|=1$ for all θ, and $|M(\theta)-M(\theta_0)| \leqq |B'(e^{i\theta}) - B'(e^{i\theta_0})|$. The continuity of $B'(e^{i\theta})$ at $\theta=\theta_0$ implies that of $M(\theta)$ at $\theta=\theta_0$.

But then $B(e^{i\theta}) = [e^{i\theta}B'(e^{i\theta})]/M(\theta)$ is also continuous at $\theta=\theta_0$, and this contradicts Theorem 2. Thus $B'(e^{i\theta})$ is discontinuous at $\theta=\theta_0$ if $e^{i\theta_0}$ is in A'.

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