LINEAR CONTINUOUS FUNCTIONALS ON THE SPACE (BV) WITH WEAK TOPOLOGIES

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In two recent papers, G. L. Krabbe [**6**], [**7**] has obtained in a general setting a theorem which for the special case when the functions involved are complex valued, yields a representation for the linear continuous functional on the space of functions of bounded variation (BV) on a finite interval with the weak topology of pointwise convergence. Some of the derivation uses the algebraic properties of the space (BV). We derive here the special case when the functions are complex valued, since it can be obtained by simple analysis methods and incidentally brings out relations between modifications of the Stieltjes integral definition which allow for the existence of the integral when the two functions involved have simultaneous discontinuities. The derivation also suggest the determination of a corresponding representation when the weak topology on (BV) is strengthened to uniform convergence.

1. **The space** (BVW). Suppose that (BVW) is the class of complex valued functions $g(x)$ of bounded variation on the finite closed interval $[a, b]$ with weak topology induced by $\lim_p g_p(x) = g(x)$ pointwise for any directed set P of p, with $V(g_p; a, b)$ bounded in p. Let $T(g)$ be a linear functional on (BVW) to the complex numbers, continuous in this topology. To determine the form of $T(g)$ we proceed in the usual way [**5**, p. 702]. Let $\gamma(t, x) = 1$ for $a \leqq x \leqq t \leqq b$, and $= 0$ for $a \leqq t < x \leqq b$, and $\delta(t, x) = 1$ for $x = t$ and $= 0$ for $x \neq t$. Then as is well known [**8**, p. 59] the space (BVW) is contained in the linear closed extension of the systems of functions $\gamma(t, x)$ and $\delta(t, x)$ with t in $[a, b]$. In particular if $\sigma \equiv a = t_0 < t_1 < \cdots < t_n = b$ is a subdivision of $[a, b]$, and we define:

$$g_\sigma(x) = g(a)\gamma(a, x) + \sum_{i=1}^{n} [g(t_i')(\gamma(t_i, x) - \gamma(t_{i-1}, x) + (g(t_i) - g(t_i'))\delta(t_i, x)]$$

with $t_{i-1} < t_i' < t_i$, then for all σ: $V(g_\sigma; a, b) \leqq V(g; a, b)$ and $\lim_\sigma g_\sigma(x) = g(x)$ for x on $[a, b]$, where the limit is directed by successive subdivisions and convergence is actually uniform in x. We note that in the topology of (BVW): $\lim_{t\to t_0+0} \gamma(t, x) = \gamma(t_0, x)$ and $\lim_{t\to t_0-0} \gamma(t, x)$

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$=\gamma(t_0, x)-\delta(t_0, x)$. If we define $T(\gamma(t, x))=f(t)$ and $T(\delta(t, x))=h(t)$, then $f(t+0)=f(t)$ for t on $[a, b)$ and $f(t-0)=f(t)-h(t)$ for t on $(a, b]$, that is $f(t)$ is continuous on the right and has a limit on the left throughout $[a, b]$ or is quasi-continuous (QC). Further:

$$
\begin{aligned}
T(g) &= \lim_{\sigma} T(g_\sigma) \\
&= g(a)f(a) + \lim_{\sigma} \sum_{i=1}^{n} [g(t_i')(f(t_i) - f(t_{i-1})) \\
&\qquad\qquad + (g(t_i) - g(t_i'))(f(t_i) - f(t_i - 0))] \\
&= g(a)f(a) + \lim_{\sigma} \sum_{i=1}^{n} [g(t_i')(f(t_i - 0) - f(t_{i-1})) \\
&\qquad\qquad + g(t_i)(f(t_i) - f(t_i - 0))].
\end{aligned}
$$

Since $f(t+0)=f(t)$ for t on $[a, b)$, the sum term agrees with the approximating sum term for the Young-Stieltjes (Y) integral [**3**, p. 88], viz.

$$\sum_{i=1}^{n} [g(t_i')(f(t_i - 0) - f(t_{i-1} + 0)) + g(t_i)(f(t_i + 0) - f(t_i - 0))].$$

Consequently: $T(g)=g(a)f(a)+(Y)\int_a^b g df$, where the integral exists as the σ-integral by successive subdivisions. If we invoke the integration by parts theorem of J. S. MacNerney [**9**], which connects the (Y)-integral with the interor (I)-integral (also called modified or Dushnik integral [**3**, p. 96], [**4**, p. 273]) based on the approximating sums

$$\sum_{i=1}^{n} f(t_i')(g(t_i) - g(t_{i-1})) \quad \text{with} \quad t_{i-1} < t_i' < t_i,$$

then we have:

$$T(g) = f(b)g(b) - (I)\int_a^b f dg.$$

Other forms can be obtained if we make use of the fact that any function of bounded variation $g(x)$ is the sum of its continuous part $g_c(x)$ and its function of the breaks $g_b(x)$. This gives

$$
\begin{aligned}
(I) \int f dg &= \int f dg_c + (I) \int f dg_b \\
&= \int f dg_c + \sum_x [f(x-0)(g(x)-g(x-0)) + f(x+0)(g(x+0)-g(x))],
\end{aligned}
$$

where here and in the sequel the interval of integration is $[a, b]$, the first integral on the right is an ordinary Riemann-Stieltjes integral, and the sum term is an absolutely convergent series. Now for the Left Cauchy (LC) integral based on the sum of terms $f(x_{i-1})(g(x_i) - g(x_{i-1}))$, we have for $f(x)$ in (QC) and $g(x)$ in (BV):

$$(LC) \int fdg = \int fdg_c + \sum_x [f(x-0)(g(x) - g(x-0)) + f(x)(g(x+0) - g(x))].$$

Consequently if f is right continuous, then

$$(I) \int fdg = (LC) \int fdg.$$

If we use the integration by parts theorem which connects the (LC) and Right Cauchy (RC) integral we get:

$$T(g) = g(b)f(b) - (I) \int_a^b fdg = f(b)g(b) - (LC) \int_a^b fdg = f(a)g(a) + (RC) \int_a^b gdf.$$

As a consequence we have:

THEOREM I. *If $T(g)$ is a linear continuous functional on the space* (BVW), *then there exists a right continuous function $f(x)$ in* (QC) *such that*

$$T(g) = f(a)g(a) + (Y) \int_a^b gdf = f(b)g(b) - (I) \int_a^b fdg = f(b)g(b) - (LC) \int_a^b fdg = f(a)g(a) + (RC) \int_a^b gdf.$$

Here the second and fourth representations agree with those obtained by G. L. Krabbe [**6**, p. 188], [**7**, p. 58].

It remains to show that every such expression defines a linear functional continuous in the topology of (BVW) for every right continuous function f in (QC). For this purpose we prove the following extension of the Helly-Bray convergence theorem on Stieltjes integrals [**2**, p. 288], [**1**, p. 180]:

THEOREM II. *If $g_p(x)$ is a directed set of functions of bounded varia-*

tion with $V(g_p;\, a,\, b)$ *bounded in* p, *and* $\lim_p g_p(x)=g(x)$ *for all* x *on* $[a,\, b]$, *then* $\lim_p\,(I)\int f dg_p=(I)\int f dg$ *for all functions in* (QC); *further* $\lim_p (LC)\int f dg_p=(LC)\int f dg\,[\lim_p (RC)\int f dg_p=(RC)\int f dg]$ *for all functions* f *in* (QC) *continuous on the right* [*left*], *the interval of integration being* $[a,\, b]$ *throughout.*

There are two ways of proving this theorem. Following H. E. Bray [**1**, p. 180], we note that if σ is any subdivision of a, b, then:

$$\left|(I)\int_a^b f dg-\sum_{i=1}^n f(x_i')(g(x_i)-g(x_{i-1}))\right|\leqq\sum_{i=1}^n\omega(f;\,(x_{i-1},\,x_i))V(g;\,x_{i-1},x_i)$$

where $\omega(f;\,(x_{i-1},\,x_i))$ is the oscillation of f on the open interval $(x_{i-1},\,x_i)$. This follows from the definition of the integral and its additive property as a function of intervals. Now if f is in (QC), then as is well known [**8**, p. 59] there exists a σ_e such that for $\sigma\geqq\sigma_e$, we have $\omega(f;\,(J))\leqq e$, for all open intervals (J) of σ. Hence if $V(g_p;\,a,\,b)\leqq M$ and $\sigma\geqq\sigma_e$, then

$$\left|(I)\int_a^b f dg_p-\sum_{i=1}^n f(x_i')(g_p(x_i)-g_p(x_{i-1}))\right|\leqq eM,$$

for all p, i.e. $(I)\int f dg_p$ exists uniformly in p. Since for fixed σ, and fixed x_i' points we have $\lim_p\sum f\Delta g_p=\sum f\Delta g$, we can apply the iterated limits theorem [**3**, p. 14] and obtain $\lim_p\,(I)\int f dg_p=(I)\int f dg$.

The same procedure works for the second statement in the theorem, excepting that in this case we have for any subinterval $[c,d]$ of $[a,b]$ and g in (BV):

$$\left|(LC)\int_c^d f dg-f(c)(g(d)-g(c))\right|\leqq\omega(f;\,[c,d))V(g;\,c,d))$$

where the oscillation ω of f is on the half open interval $[c,\,d)$. The right continuity of f at each point of $[a,\,b)$ together with the fact that $f(x-0)$ exists for each x of $(a,\,b]$ now enables one to obtain for $e>0$ subdivisions σ_e such that for $\sigma\geqq\sigma_e$ and for each half open interval $[c,\,d)$ in σ one has $\omega(f;\,[c,\,d))<e$.

The situation for the (RC) integrals follows by parity.

For an alternate proof, we use the fact that if f is in (QC) then there exist sequences of step functions $\{f_n\}$ such that $\lim_n f_n=f$ uniformly on $[a,\,b]$. Now if $\lim_n f_n=f$ uniformly on $[a,\,b]$ and $V(g_p;\,a,\,b)$ is bounded in p, then $\lim_n\int f_n dg_p=\int f dg_p$ uniformly in p for the integrals considered, since for each of them we have $\left|\int f_n dg_p-\int f dg_p\right|\leqq\|f_n-f\|\,V(g_p;\,a,\,b)$, $\|f\|$ being the least upper bound norm. If f_n is a step function with steps at $a=x_0<x_1<\cdots<x_m=b$, then

$$(I) \int_a^b f_n dg_p = \sum_{i=1}^n f_n(x_i')(g_p(x_i) - g_p(x_{i-1})), \quad \text{with} \quad x_{i-1} < x_i' < x_i.$$

Consequently $\lim_p (I)\int f_n dg_p = (I)\int f_n dg$ for all n. The iterated limits theorem now gives:

$$\lim_p \lim_n \int f_n dg_p = \lim_n \lim_p \int f_n dg_p \text{ or} \lim_p (I) \int f dg_p = (I) \int f dg.$$

For the case of the (LC) integral we note that if $f(x)$ is a step function continuous on the right then:

$$(LC) \int_a^b f dg = \sum_{i=1}^m f(x_{i-1})(g(x_i) - g(x_{i-1})),$$

where x_i, $i=0, \cdots, m$ are the steps. Then the above reasoning is effective here also.

If we wish Theorem II to hold for all functions in (QC) then additional conditions must be added. We have:

COROLLARY. *If we add to the main hypothesis of Theorem* II: (a) *the condition*: $\lim_p g_p(x-0) = g(x-0)$ *for* x *on* $(a, b]$, *then* $\lim_p (RC)\int f dg_p = (RC)\int f dg$; (b) *the condition*: $\lim_p g_p(x+0) = g(x+0)$ *for* x *on* $[a, b)$, *then* $\lim_p (LC)\int f dg_p = (LC)\int f dg$; *and* (c) *the conditions* (a) *and* (b), *then* $\lim_p (Y)\int f dg_p = (Y)\int f dg$, *each for all* f *in* (QC). *Condition* (c) *will obviously hold if the convergence of* $g_p(x)$ *to* $g(x)$ *is uniform on* $[a, b]$.

This corollary follows easily from the second method of proof since the expressions for the integrals when f is a step function involve $g(x-0)$ for the (RC) integral, $g(x+0)$ for the (LC) integral and both for the (Y) integral. The first method of proof does not seem to be applicable.

In view of Theorem II, the equivalent expressions for $T(g)$ in Theorem I define linear continuous functionals in the topology of (BVW), so that each form gives a representation for $T(g)$. Theorem II also shows that an expression of the form $Ag(a)+Bg(b)+(I)\int_a^b f dg$ defines a linear continuous functional on (BVW) for all f in (QC), while $Ag(a)+Bg(b)+(RC)\int_a^b f dg$ is a linear continuous functional on (BVW) for all left continuous functions f in (QC). These can obviously be expressed in the forms of Theorem I by using a different f.

An examination of the proofs of the theorems indicates that the same type of reasoning yields corresponding results for the case when $T(g)$ is a linear transformation on (BVW) to a linear normed complete space, or even a linear complete space with topology based on semi-norms.

2. **The space** (BVWU). The fact that the approximations by step functions to functions of bounded variation converge uniformly on the interval $[a, b]$ suggests the question: What is the form of a linear continuous functional on the space (BV) when the topology is based on convergence: $\lim_p g_p(x) = g(x)$ uniformly on $[a, b]$ for any directed set P of p, with $V(g_p; a, b)$ bounded in p. Let us call this space (BVWU).

To obtain the form of such a functional we can proceed as in Theorem I, which yields:

THEOREM III. *If $T(g)$ is a linear continuous functional on* (BVWU) *to real numbers, then there exist functions $f(x)$ and $h(x)$ such that*

$$T(g) = (I) \int_a^b g df + \sum_x (g(x) - g(x-0))h(x)$$

where the (I)-integral exists as a σ-integral and the infinite series $\sum_x$ converges absolutely, each for all functions g in (BV). *The function $f(x)$ is in* (QC) *and the function $h(x)$ is zero except for a sequence $\{x_n\}$ such that* $\lim_n h(x_n) = 0$.

The last sentence is the only part of this theorem which requires demonstration.

Suppose, if possible, that $f(t) = T(\gamma(t, x))$ is not in (QC). Then there exists $e > 0$, an x_0 in $[a, b]$ and a sequence of disjoint intervals (x_n', x_n'') converging to x_0 from one side, such that $|f(x_n') - f(x_n'')| > e$. Let

$$g_n(x) = \frac{1}{n} \sum_{m=n+1}^{2n} \operatorname{sgn}(f(x_n'') - f(x_n'))(\gamma(x_n'', x) - \gamma(x_n', x)).$$

Then $V(g_n; a, b) = 2$, $\lim_n g_n(x) = 0$ uniformly on $[a, b]$ but $T(g_n(x)) > e$ for all n.

Similarly suppose that for $h(t) = T(\delta(t, x))$ there exists $e > 0$ and an infinite sequence $\{x_n\}$ such that $|h(x_n)| > e$ for all n. Set

$$g_n(x) = \frac{1}{n} \sum_{m=n+1}^{2n} \operatorname{sgn}(h(x_n))\delta(x_n, x).$$

Then $V(g_n; a, b) = 2$, $\lim_n g_n(x) = 0$ uniformly but $T(g_n(x)) > e$ for all n. Hence the set x_n for which for a given $e > 0$, $|h(x_n)| > e$ is finite.

To complete the derivation for $T(g)$ we prove:

THEOREM IV. *If $f(x)$ is in* (QC) *and $h(x)$ vanishes except for a sequence $\{x_n\}$ such that* $\lim_n h(x_n) = 0$, *then* $T(g) = (I)\int_a^b g df + \sum_x (g(x) - g(x-0))h(x)$ *is a linear continuous functional on* (BVWU).

The linear property is obvious. For the continuity property we note that since f is in (QC) integration by parts [**9**] applies, i.e.

$$(I)\int_a^b g\,df = g(b)f(b) - g(a)f(a) - (Y)\int_a^b f\,dg.$$

Then the Corollary to Theorem II applied to the (Y) integral yields the desired continuity under uniform convergence in (BV).

As for the continuity of the expression $\sum_x (g(x)-g(x-0))h(x)$ this is an immediate consequence of the well known weak continuity theorem in the space l^1 of sequences, viz. if $\{a_{pn}\}$ are in l^1 for each p, with $\sum_n |a_{pn}|$ bounded in p, if $\lim_p a_{pn}=a_n$ for each n and if $\{y_n\}$ is in c_0, i.e. $\lim_n y_n=0$, then $\lim_p \sum_n a_{pn}y_n = \sum_n a_n y_n$.

We note that the condition that $V(g_p;\, a,\, b)$ is bounded in p in the weak convergence conditions can be replaced by the condition that $V(g_p;\, a,\, b)$ is ultimately bounded in p, i.e. there exists p_0 and M such that if $p \geqq p_0$ then $V(g_p;\, a,\, b) \leqq M$. Also, only slight changes are necessary if we assume the linear functionals complex valued.

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