METRIC ENTROPY OF CERTAIN CLASSES OF LIPSCHITZ FUNCTIONS

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1. Introduction. In this paper we discuss the metric entropy (in the uniform metric) of certain classes Lip(α/A) of Lipschitz functions which will be defined in §2.

The notion of metric entropy (or ε-entropy) of a totally bounded subset A of a metric space was introduced by Kolmogorov [1] to characterize the massiveness of A. Among the most striking applications of this notion are the results of Kolmogorov [3] and Vitushkin [7]; for expositions of the subject of metric entropy see Lorentz [4], [5].

We collect some basic definitions and facts. A will always denote a nonempty subset of a metric space X, and ε a positive number not exceeding unity. We use the notation f(ε) ~ g(ε) to mean \( \lim (f(ε)/g(ε)) = 1 \) as ε \( \to 0^+ \) and f(ε) \( \ll g(ε) \) to mean \( f(ε) = O(g(ε)) \) as ε \( \to 0^+ \); f(ε) \( \approx g(ε) \) means that both f(ε) \( \ll g(ε) \) and g(ε) \( \ll f(ε) \). All logarithms will have base 2.

Definition 1. A class C of subsets of X is called an ε-cover of A if each set in C has diameter not exceeding 2ε and A \( \subseteq \bigcup \{ C : C \subseteq C \} \).

Definition 2. A subset D of X is called ε-distinguishable if the distance between each pair of points of D exceeds ε.

Definition 3. A subset N of X is called an ε-net for A if each point of A is within distance ε of some point of N.

For totally bounded sets A (i.e., sets having a finite ε-cover for each ε \( > 0 \)) we make the following definitions which are due to Kolmogorov [3].

Definition 4. \( N_ε(A) \) denotes the minimal number of sets in any ε-cover of A.

Definition 5. \( M_ε(A) \) denotes the maximal number of points in any ε-distinguishable subset of A.

Definition 6. \( H_ε(A) = \log N_ε(A) \) is called the ε-entropy of A.

Definition 7. \( C_ε(A) = \log M_ε(A) \) is called the ε-capacity of A.

The following basic theorem of Kolmogorov [3, §1] will be used.

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Theorem. Let $A$ be a totally bounded subset of a metric space $X$, $\epsilon > 0$. Then

$$(1) \quad M_{2\epsilon}(A) \leq N_\epsilon(A) \leq M_\epsilon(A),$$

and hence also

$$(2) \quad C_{2\epsilon}(A) \leq H_\epsilon(A) \leq C_\epsilon(A).$$

It should also be noted that for $A \subset (-\infty, +\infty)$

$$(3) \quad N_{\epsilon}(A) \leq TN_{\epsilon}(A)$$

holds for each integer $T \geq 1$ (each set which covers $A$ and has diameter $\leq T\epsilon$ can be replaced by $T$ sets of diameter $\leq \epsilon$).

2. The entropy of Lip $(\alpha/A)$. In the following $A$ will always denote a compact subset of $[0, 1]$. All function spaces considered will have the uniform metric.

Definition 8. Let $\alpha > 0$ and let $A$ be a compact subset of $[0, 1]$. By Lip $(\alpha/A)$ is meant the (compact) set of all real valued functions defined on $A$ for which $|f(x) - f(x')| \leq |x - x'|^\alpha$ for all $x, x'$ in $A$ and $\max_{x \in A} |f(x)| \leq 1$.

It will be convenient to consider Lip $(\alpha/A)$ as a subspace of $M(\alpha/A)$, the space of all functions on $A$ for which $|f(x) - f(x')| \leq 5|x - x'|^\alpha$ for all $x, x'$ in $A$ and $\max_{x \in A} |f(x)| \leq 2$. We investigate how $H_\epsilon$(Lip $(\alpha/A)$) depends upon $A$. For the case $A = [0, 1]$, Kolmogorov and Tihomirov [3] have shown

$$(4) \quad H_\epsilon$(Lip $(\alpha/A)$) $\approx \epsilon^{-1/\alpha}.$$

Also, it is easy to establish [3, §9(235), p. 354] that for any non-empty totally bounded set $B$ in a metric space (letting $\delta = \epsilon^{1/\alpha}$)

$$(5) \quad H_\epsilon$(Lip $(\alpha/B)$) $\gg N_{\delta}(B) + \log \epsilon^{-1}.$$

Related estimates which will not be needed here can be found in [6] and [8, §17].

In the following we obtain an upper estimate for $H_\epsilon$(Lip $(\alpha/A)$) which leads to a simple characterization of compact subsets $A$ of $[0, 1]$ for which (4) holds.

Theorem 1. Let $A$ be a compact subset of $[0, 1]$, and assume $0 < \alpha \leq 1$. Then, letting $\delta = \epsilon^{1/\alpha}$, we have

$$(6) \quad H_\epsilon$(Lip $(\alpha/A)$) $\ll N_{\delta}(A) \cdot \log(2\epsilon^{-1}(N_{\delta}(A))^{-\alpha}) + \log \epsilon^{-1}.$$

Proof. Let $\epsilon (0 < \epsilon < 1)$ be given and let $A_0 = \{x_1, x_2, \ldots, x_M\}$ be (from left to right) a maximal set of $\delta/2$-distinguishable points of
A (M abbreviates $M_{\delta/2}(A)$, and we can assume $M \geq 2$), and for each $i=1, 2, \ldots, M$, let $I_i$ be the closed interval centered at $x_i$ having length $\delta$. The class $\{I_i: i=1, 2, \ldots, M\}$ covers $A$, since otherwise $A_0$ would not be maximal.

Let $F_0$ denote the family of functions obtained by restricting the functions of $\text{Lip}(\alpha/A)$ to $A_0$. Further, for each $h \in F_0$, let $f_h$ be a function defined on $A_0$ such that $f_h(x_i)$ is an integral multiple of $\varepsilon$ and $|f_h(x_i) - h(x_i)| \leq \varepsilon$ for each $i=1, 2, \ldots, M$. Let $F^*$ denote the set of functions so obtained. It is easy to see that $F^* \subseteq M(\alpha/A_0)$. In fact, letting $f_h$ be an arbitrary function of $F^*$ we have, for any $x_i, x_j$ in $A_0$, 

$$|f_h(x_i)| \leq |f_h(x_j)| + \varepsilon \leq 1 + 1$$

and $|f_h(x_i) - f_h(x_j)| \leq |h(x_i) - h(x_j)| + 2\varepsilon = |h(x_i) - h(x_j)| + 2\delta \alpha \leq |x_i - x_j|^\alpha + 2\delta \alpha \leq |x_i - x_j|^\alpha + 2^{\alpha+1}|x_i - x_j|^\alpha \leq |x_i - x_j|^\alpha$; hence $F^* \subseteq M(\alpha/A_0)$. Further let $F$ denote a subset of $M(\alpha/A)$ obtained by extending each function of $F^*$ to $A$. It is easy to see that such an $F$ exists (since $A_0$ is finite, each function of $M(\alpha/A_0)$ can be extended linearly to be a function of $M(\alpha/A)$).

Now let $n(F)$ denote the number of elements of $F$. Then $n(F) \geq N_{14\varepsilon}(\text{Lip}(\alpha/A))$, which can be seen as follows. Let $g \in \text{Lip}(\alpha/A)$ and let $f$ be a function of $F$ which at each $x_i$ ($i=1, 2, \ldots, M$) has values no farther than $\varepsilon$ from those of $g$. Let $a \in A$, so $a \in I_i$ for some $i=1, 2, \ldots, M$. Then we have $|f(a) - g(a)| \leq |f(a) - f(x_i)| + |f(x_i) - g(x_i)| + |g(x_i) - g(a)| \leq 5(\varepsilon^1)^\alpha + \varepsilon + (\varepsilon^{1/\alpha})^\alpha = 7\varepsilon$. Hence $\|f - g\| = \max_{a \in A} |f(a) - g(a)| \leq 7\varepsilon$, showing that $F$ is a $7\varepsilon$-net for $\text{Lip}(\alpha/A)$.

Since the family of spheres of diameter $14\varepsilon$ centered at the points of $F$ covers $\text{Lip}(\alpha/A)$, $N_{14\varepsilon}(\text{Lip}(\alpha/A))$ cannot exceed $n(F)$.

It remains to estimate $n(F)$ from above. To do this let $f \in F$ and note that there are no more than $[4/\varepsilon]+1$ possible values of $f(x_1)$; for each of these there are no more than $[2(x_2 - x_1)^{\alpha \varepsilon^{-1}}] + 1^2$ possible values of $f(x_2)$, and, in general, for each $k = 1, 2, \ldots, M - 1$, there are no more than $[2(x_{k+1} - x_k)^{\alpha \varepsilon^{-1}}] + 1$ possible values of $f(x_{k+1})$. Furthermore, for each $k = 1, 2, \ldots, M - 1$, we have $[2(x_{k+1} - x_k)^{\alpha \varepsilon^{-1}}] + 1 \leq 4(x_{k+1} - x_k)^{\alpha \varepsilon^{-1}}$ because $2x + 1 \leq 4x$ if $x \geq 1/2$ (in our case $x = (x_{k+1} - x_k)^{\alpha \varepsilon^{-1}} \geq 1/2$ because $A_0$ is $(1/2)\varepsilon^{1/\alpha}$-distinguishable and hence, since $0 < \alpha \leq 1$, $(\varepsilon/2)^{1/\alpha}$-distinguishable). Thus $n(F)$ does not exceed

$$\left([4/\varepsilon] + 1\right)4^{M-1}e^{-\varepsilon^{-1}} \prod_{k=1}^{M-1} (x_{k+1} - x_k)^\alpha.$$  

\[ (7) \]

And since any product $y_1 \cdot y_2 \cdot \cdots \cdot y_{M-1}$ subject to the conditions $y_i > 0$ ($i=1, 2, \ldots, M-1$) and $\sum_{i=1}^{M-1} y_i = \text{constant}$ is maximized by

\[ [x] \text{ denotes the largest integer not exceeding } x. \]
taking \( y_1 = y_2 = \cdots = y_{M-1} \), (7), and hence also \( N_\alpha(A) \), does not exceed \( 5\epsilon^{-1}(4\epsilon^{-1}(M-1)^{-\alpha})^M \leq 5\epsilon^{-1}(8\epsilon^{-1}M^{-\alpha})^M \).

Thus, letting \( N \) abbreviate \( N_{\frac{\delta}{2}}(A) \) and using (1) and (3),

\[
H_\alpha(A) \leq \log 5\epsilon^{-1} + M \log(8\epsilon^{-1}M^{-\alpha})
\leq \log 5\epsilon^{-1} + N_{\frac{\delta}{4}}(A) \log(8\epsilon^{-1}N^{-\alpha})
\ll N_{\delta}(A) \cdot \log(2\epsilon^{-1}(N_{\delta}(A))^{-\alpha}) + \log \epsilon^{-1}
\]

follows immediately, which concludes the proof of the theorem.

The maximum possible value of \( N_\epsilon(A) \) for sets \( A \subset [0, 1] \) occurs when \( A = [0, 1] \), in which case \( \epsilon^{-1} \leq N_\epsilon(A) \leq \epsilon^{-1} + 1 \); the relation \( N_\epsilon(A) = o(\epsilon^{-1}) \) means that \( A \) is in some sense rarified in \([0, 1]\). Likewise, the maximum possible value of \( H_\epsilon(A) \) for \( A \subset [0, 1] \) is \( H_\epsilon(A) = \epsilon^{-1/\alpha} \). The following theorem shows that for sets \( A \) which are rarified in \([0, 1]\), \( H_\epsilon(A) \) cannot reach its maximum; it also characterizes when this maximum is achieved. First we need the following lemma.

**Lemma 1.** Let \( A \) be a compact subset of \([0, 1]\).

(i) If \( \text{meas } A > 0 \) then \( N_\epsilon(A) \approx \epsilon^{-1} \).

(ii) If \( \text{meas } A = 0 \) then \( N_\epsilon(A) = o(\epsilon^{-1}) \).

**Proof.** For any interval \( I \) of length \( l \) we have \( l/(2\epsilon) \leq N_\epsilon(A) < l/(2\epsilon) + 1 \). From the definitions of \( N_\epsilon(A) \) and Lebesgue measure

\[
\text{meas}(A)/(2\epsilon) \leq N_\epsilon(A) \leq N_\epsilon[0, 1] < 1/(2\epsilon) + 1.
\]

This proves (i).

To prove (ii), suppose \( \text{meas } A = 0 \) and let \( \delta > 0 \) be arbitrary. Then there exist finitely many intervals \( I_i \) of lengths \( l_i, \ i = 1, 2, \cdots, k \), which cover \( A \) and for which \( \sum_{i=1}^{k} l_i \leq \delta \). Letting \( A_i = A \cap I_i \), we have

\[
N_\epsilon(A) \leq \sum_{i=1}^{k} N_\epsilon(A_i) \leq \sum_{i=1}^{k} (l_i/2\epsilon + 1) \leq \delta/2\epsilon + k < \delta/\epsilon
\]

for all sufficiently small \( \epsilon > 0 \), which concludes the proof of the lemma.

**Theorem 2.** Let \( A \) be a compact subset of \([0, 1]\). If \( \text{meas}(A) > 0 \), then

\[
H_\epsilon(A) \approx \epsilon^{-1/\alpha};
\]

if \( \text{meas}(A) = 0 \), then

\[
H_\epsilon(A) = o(\epsilon^{-1/\alpha}).
\]

**Proof.** Let \( \delta = \epsilon^{1/\alpha} \). If \( \text{meas}(A) > 0 \), then Lemma 1 yields \( N_{\delta}(A) \approx 1/\delta \). Using (5) and (6) this yields \( H_\epsilon(A) \approx N_{\delta}(A) \log \epsilon^{-1} \approx \epsilon^{-1/\alpha} \).
If \( \text{meas}(A) = 0 \), then for \( T_* = \delta N_8(A) \) we have, by Lemma 1, \( T_* = o(1) \).

Therefore, by (6),

\[
H_*(\text{Lip}(\alpha/A)) \ll \delta^{-1} T_* \log(2(T_*)^{-\alpha}) + \log \varepsilon^{-1} = o(\delta^{-1}) + \log \varepsilon^{-1} = o(\varepsilon^{1/\alpha}),
\]

which completes the proof of the theorem.

**References**


2. A. N. Kolmogorov and V. M. Tihomirov, *\( \varepsilon \)-entropy and \( \varepsilon \)-capacity of sets in function spaces*, Uspehi Mat. Nauk 14 (1959), 3–86.


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