ČEBYŠEV SUBSPACES OF FINITE DIMENSION IN L_1

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1. **Introduction.** If E is a real normed linear space, a subspace M of E is said to be a *Čebyšev subspace* if to each y in E there exists a unique nearest element x in M, that is, an element x in M such that $\|y-x\|<\|y-z\|$ for $z\in M$, $z\neq x$. The problem of characterizing such subspaces for the classical Banach spaces of functions is an interesting one. The first (and best) result of this nature was the classical theorem of Haar [6] characterizing the finite dimensional Čebyšev subspaces of $C[0, 1]$. Much recent work has been done on this and related questions by A. L. Garkavi [3], [4], [5], [6] and the author [9], [10], stimulated in part by the discovery [9] of a certain duality between uniqueness of best approximation and uniqueness of Hahn-Banach extensions of functionals. So far, however, there has been no characterization given of the finite dimensional Čebyšev subspaces of $L_1(T, \Sigma, \mu)$ (T σ-finite). This is partly due to the fact that if (T, Σ, μ) is nonatomic, then $L_1(T, \Sigma, \mu)$ contains *no* finite dimensional Čebyšev subspaces, a result due to M. S. Krein and B. Ya. Levin [1, Chapter IV] for $T=[0, 1]$, μ Lebesgue measure, and to Henry Dye [9] in the general case. In fact, Garkavi [4] has extended these arguments to show that $L_1(T, \Sigma, \mu)$ *contains a Čebyšev subspace of dimension* $n<\infty$ *if and only if* (T, Σ, μ) *contains at least* n *atoms*. This result, of course, does not tell *which* subspaces are Čebyšev subspaces, a question which we consider in the present paper Our characterization is not entirely satisfactory, in that it is given in terms of the elements of the annihilator $M^{\perp}$ of M in L_∞, rather than in terms of the elements of M itself, but we see no way to avoid this. ($M^{\perp}=\{f: f\in L_\infty(T, \Sigma, \mu)$ and $\int_T fg\,d\mu=0$ for each $g\in M\}$.) The proof makes essential use of an interpolation theorem (Lemma 2) which is actually a reformulation of Liapunov's theorem on the range of a vector measure.

The theorem itself has two corollaries of interest; one immediate, the other not so immediate. The first one is simply the restatement of the theorem for the case when $L_1(T, \Sigma, \mu)$ is actually the space l_1.

COROLLARY 1. *Suppose that M is an n-dimensional subspace of l_1. Then M is a Čebyšev subspace if and only if for each $z=\{z_k\}$ in $M^{\perp}\subset l_\infty$, $z\neq 0$, there are at least n integers k such that $|z_k|<\|z\|_\infty$.*

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The second corollary characterizes the subspaces M of finite codimension in $E=C(T)$ (T compact Hausdorff) which have *property* U: Every continuous linear functional on M has a unique norm-preserving extension to E. The relation between this property and the Čebyšev property arises from the fact [9] that *a subspace M of a normed linear space E has property U if and only if $M^{\perp}$ is a Čebyšev subspace of E^*.* Again, our characterization is not as "intrinsic" as we would like, but an example shows that the only obvious improvement is impossible.

If $f \in L_\infty(T, \Sigma, \mu)$, denote by $S(f)$ the measurable set (defined to within a set of measure zero) on which $|f| < \|f\|$. Our main theorem is the following:

THEOREM 1. *Suppose that M is an n-dimensional subspace of $L_1(T, \Sigma, \mu)$. Then M fails to be a Čebyšev subspace if and only if there exists f in $M^{\perp} \subset L_\infty(T, \Sigma, \mu)$, $f \neq 0$, such that $S(f)$ is purely atomic and contains at most $n-1$ atoms.*

The proof of Theorem 1 depends on several lemmas, the first of which actually characterizes Čebyšev subspaces of arbitrary dimension in L_1. The lemma itself is new, but the ideas in the proof have all been used previously.

For a measurable function g, let $Z(g)=\{t: g(t)=0\}$; for elements of L_1 this is defined only to within a set of measure zero, and set theoretic operations with both $Z(g)$ and $S(f)$ are to be interpreted to be modulo sets of measure zero.

LEMMA 1. *A closed linear subspace M of L_1 is not a Čebyšev subspace if and only if there exist f in $M^{\perp}$, $\|f\|=1$, and g in M, $g \neq 0$, such that $S(f) \subset Z(g)$.*

PROOF. If M is not a Čebyšev subspace there exist h in $L_1 \sim M$ and g in M, $g \neq 0$, such that $1=\|h\|=\|h+g\|=d(h, M)$. By the Hahn-Banach theorem there exists f in $M^{\perp} \subset L_\infty$ such that $\|f\|=1=\int fh\,d\mu = \int f(h+g)\,d\mu$. It follows that $S(f) \subset Z(h) \cap Z(h+g) \subset Z(g)$.

To prove the converse, suppose that f and g exist as above, and define $h=f|g|$. We will show that $\int fh\,d\mu = \|h\|$ and that $\|h\|=\|h-g\|$. (Since, for any g' in M, $\|h-g'\| \geqq \int f(h-g')\,d\mu = \int fh\,d\mu = \|h\|$, this will show that h has 0 and g as distinct nearest points in M.) First,

$$\int fh\,d\mu = \int f^2 |g|\,d\mu = \int |g|\,d\mu = \int |h|\,d\mu = \|h\|.$$

Second, we note that

$$|h-g| = |(f|g| - g)| = f(f|g| - g) = |g| - fg,$$

so that

$$\|h-g\| = \int |h-g|\,d\mu = \int (|g| - fg)d\mu = \int |g|\,d\mu = \|h\|,$$

and the proof is complete.

We can now easily prove half of Theorem 1. Suppose that M is an n-dimensional subspace of L_1 and that there exists $f \neq 0$ in $M^\perp \subset L_\infty$ such that $S(f)$ is the union of k atoms, $0 \leqq k \leqq n-1$. Let $N = \{g : g \in L_1$ and $g = 0$ a.e. on $S(f)\}$. Evidently N has finite codimension $k < n$, so that there exists $g \neq 0$ in $M \cap N$, and hence M is not a Čebyšev subspace, by Lemma 1.

The proof of the converse is more involved. If M is an n-dimensional subspace of L_1 which is not a Čebyšev subspace, then there exist f in $M^\perp$, $\|f\| = 1$, and g in M, $g \neq 0$, such that $S(f) \subset Z(g)$. We will first replace f by a function f' in $M^\perp$, $\|f'\| = 1$, for which $S(f') \subset S(f)$ and $S(f')$ is purely atomic; after that, a further replacement will yield f'' in $M^\perp$, $\|f''\| = 1$, such that $S(f') \supset S(f'')$ and the latter has at most $n-1$ atoms. To this end, write $S(f) = S \cup A$, where S contains no atoms, and A is purely atomic. (Since T is σ-finite, A is the union of at most countably many atoms.) The function f' will be defined to be equal to f on $T \sim S$ and to have absolute value 1 on S, so that $S(f') \subset A$. To this end, choose $g_2, \cdots, g_n$ such that $g_1 = g, g_2, \cdots, g_n$ form a basis for M. If we can produce f' with $|f'| = 1$ a.e. on S, $f' = f$ on $T \sim S$ and $\int_S f' g_i d\mu = \int_S f g_i d\mu$, $i = 2, \cdots, n$, then we will certainly have $\int_T f' g_i d\mu = \int_T f g_i d\mu = 0$ for each i, so that $f' \in M^\perp$. Since we are only concerned with defining f' on S, the next lemma shows that such a function exists.

LEMMA 2. *Suppose that the measure space (S, Σ, μ) contains no atoms, that f is a measurable function on S with $|f| \leqq 1$ a.e. and that $g_1, g_2, \cdots, g_k$ are in $L_1(S, \Sigma, \mu)$. Then there exists a measurable function f' on S with $|f'| = 1$ a.e. such that $\int_S f' g_i d\mu = \int_S f g_i d\mu$, $i = 1, 2, \cdots, k$.*

PROOF. Such a function f' will exist if and only if there exists a measurable set B (which will be the set on which $f' = 1$) such that $\int_S f g_i = \int_B g_i - \int_{S \sim B} g_i = 2\int_B g_i - \int_S g_i$, and this is equivalent to $\int_B g_i = 1/2 \int_S (1+f) g_i$, $i = 1, 2, \cdots, k$. Now, by Liapunov's theorem [7], [8] if we define, for C in Σ, $\phi(C) = (\int_C g_1, \int_C g_2, \cdots, \int_C g_k)$, then the image K (in R^k) of Σ under the map ϕ is a compact convex set. On the other hand, if we define, for $|g| \leqq 1$,

$$\psi(g) = \left(1/2 \int_S (1+g)g_1, 1/2 \int_S (1+g)g_2, \cdots, 1/2 \int_S (1+g)g_k\right),$$

then ψ is a weak* continuous affine map from the unit ball U of L_∞ into R^k, hence $\psi(U)$ is also compact and convex. Our problem is to find B in Σ such that $\phi(B)=\psi(f)$, and for this it suffices to show that $\psi(f)\in K$. This will certainly be true if $\psi(U)\subset K$, and for *this* it is sufficient that $\psi(\text{ext } U)\subset K$, where ext U is the set of extreme points of U. Now, as is well known (and easy to prove), if $g\in\text{ext } U$, then $|g|=1$ a.e., i.e. there exists C in Σ such that $g=2\chi_C-1$. It follows that $\psi(g)=(\int_C g_1, \cdots, \int_C g_k)=\phi(C)\in K$, which completes the proof of the lemma.

[It is interesting (but not pertinent to the questions at hand) to note that one can easily deduce the Liapunov theorem from this lemma. We make two observations which help in the proof. If $\mu_1, \mu_2, \cdots, \mu_n$ define a vector measure on (T, Σ), then they may be identified with their Radon-Nykodym derivatives $g_1, \cdots, g_n$ with respect to $\mu=|\mu_1|+|\mu_2|+ \cdots +|\mu_n|$. Also, if $0\leqq h\leqq 1$ is an element of $L_\infty(T, \Sigma, \mu)$, then by applying the lemma to $f=2h-1$, one obtains a set S in Σ such that $\mu_k(S)=\int hg_k d\mu$, $k=1, 2, \cdots, n$.]

It remains to define f'' so that $\|f''\|=1$, $f''\in M^\perp$, $f''=f'$ on $T\sim A$ and $|f''|=1$ on all but (at most) $n-1$ atoms in A. It suffices, then, to define f'' on A such that $|f''|=1$ on all but $n-1$ atoms and such that $\int_A f''g_i=\int_A f'g_i$, where $g_1=g, g_2, \cdots, g_n$ form a basis for M. Since all the functions involved are measurable, they are a.e. constant on atoms, so their restrictions to A may be considered to be sequences. If A has only finitely many atoms, the rest of the proof follows by induction from the next lemma, so we will assume that $A=\bigcup_{k+1}^\infty A_k$ (A_k essentially disjoint atoms) and define the obvious isometry between $L_1(A)$ and the space l_1 by $\alpha(g)_k=g(A_k)\mu(A_k)$. Thus, there is no loss in generality in assuming that the g's (which we now write as $g^1, g^2, \cdots, g^n$) are elements of l_1 and that $f'\in l_\infty$, with $|f_k'|<1$ for all k. Our problem requires us to produce f'' in l_∞ with $|f_k''|=1$ (for all but at most $n-1$ integers k) such that $\sum f_k'' g_k^i= \sum f_k' g_k^i$, $i=2, 3, \cdots, n$. We first prove a simple lemma concerning *finite* sequences.

LEMMA 3. *Suppose that $u^1, u^2, \cdots, u^{n-1}$ are elements of R^n and that $z\in R^n$, with $|z_k|<1$ for each k. Then there exists z' in R^n, with $|z_k'|\leqq 1$ for each k, $|z_k'|=1$ for at least one k, and $\sum z_k' u_k^i = \sum z_k u_k^i$, $i=1, 2, \cdots, n-1$.*

PROOF. The subspace spanned by $u^1, \cdots, u^{n-1}$ has dimension at λ most $n-1$, so there exists a in R^n, $a\neq 0$, such that $(\lambda a, u^i=0$ for in R, $i=1, 2, \cdots, n-1$. For small $\lambda\neq 0$, $|z_k+\lambda a_k|<1$, so we can choose λ_0 such that $|z_k+\lambda_0 a_k|=1$ for at least one k and $|z_j+\lambda_0 a_j|\leqq 1$ for $j\neq k$. Clearly, $z'=z+\lambda_0 a$ satisfies the above requirements.

Returning to the proof of the theorem, let $S(f')$ denote the set of integers k such that $|f_k'|<1$ and suppose $S(f')$ contains n or more integers. Write these in their natural order and consider the elements of R^n obtained by restricting f', $g^2, \cdots, g^n$ to the set S_1 consisting of the first n integers in $S(f')$. (Recall that $g^1=0$ on $S(f')$.) It follows from the above lemma (applied to these elements of R^n) that we can choose $h^1\in l_\infty$, $\|h^1\|\leqq 1$, such that $h_k^1=f_k'$ outside of S_1, $h^1\in M^\perp$, and $|h_k^1|=1$ for at least one k in S_1. Let S_2 be the first n elements of $S(h^1)$ and choose h^2 such that $\|h^2\|\leqq 1$, $h^2=h^1$ outside S_2, $h^2\in M^\perp$ and $|h_k^2|=1$ for at least one k in S_2. Continuing in this manner we obtain a sequence $\{h^j\}$ of elements of $M^\perp$ such that $\|h^j\|=1$ and for each integer k (with at most $n-1$ exceptions) there exists j_k such that $h_k^j=h_k^{j_k}$ for $j\geqq j_k$ and $|h_k^{j_k}|=1$. Since $\{h^j\}$ lies in a weak* compact set, it has a subsequence which converges to an element f'' in $M^\perp$. It follows easily that $S(f'')$ contains at most $n-1$ integers, and the proof of Theorem 1 is complete.

We can now prove our corollary concerning property U.

COROLLARY 2. *Suppose that M is a subspace of finite codimension n in $C(T)$, T compact Hausdorff. Then M fails to have property U if and only if there exists a bounded Baire measurable function f on T such that $|f|\leqq 1$, $\int_T f d\mu=0$ for each μ in $M^\perp$, and $|f(t)|<1$ for at most $n-1$ points t of T.*

PROOF. Suppose that such a function exists, and let $\mu_1, \cdots, \mu_n$ be a basis for $M^\perp$. Let $\mu=|\mu_1|+|\mu_2|+\cdots+|\mu_n|$ and write $\mu_i=f_i\mu$, where $f_i=d\mu_i/d\mu$ is the Radon-Nykodym derivative of μ_i with respect to μ. The functions f_i generate an n-dimensional subspace M_0 of $L_1(T, \Sigma, \mu)$ (Σ the Baire subsets of T) and it is clear that if M_0 is not a Čebyšev subspace of $L_1(T, \Sigma, \mu)$, then $M^\perp$ is not a Čebyšev subspace of $C(T)^*$. (Note that $L_1(T, \Sigma, \mu)$ is isometric with the subspace of measures in $C(T)^*$ which are absolutely continuous with respect to μ, and that this isometry maps M_0 onto M.) Now, $f\in L_\infty(T, \Sigma, \mu)$, $f\in M_0^\perp$, $\|f\|=1$ and $S(f)$ consists of at most $n-1$ points, so by Theorem 1, M_0 is not a Čebyšev subspace and this half of the proof is complete. To prove the converse, suppose that M is non-U, so that $M^\perp$ is not a Čebyšev subspace of $C(T)^*$. Let $\nu\in C(T)^*$, $\nu\notin M^\perp$, have two distinct nearest points in $M^\perp$, and choose a basis $\mu_1, \cdots, \mu_n$ for $M^\perp$. Let $\mu=|\nu|+|\mu_1|+\cdots+|\mu_n|$ and write $\nu=h\mu$,

$\mu_i = f_i\mu$, $i=1, \cdots, n$. Thus, the subspace M_0 in $L_1(T, \Sigma, \mu)$ generated by $f_1, \cdots, f_n$ is not a Čebyšev subspace, since h has two nearest points in M_0, so by Theorem 1 there exists a function f_0 in $L_\infty(T, \Sigma, \mu)$ of norm 1 with $\int f_0 f_i d\mu = \int f_0 d\mu_i = 0$ for $i=1, \cdots, n$ and $S(f_0)$ consists of k atoms, $0 \leqq k \leqq n-1$. Now f_0 is defined on T only to within a Baire set B of μ-measure zero. Define f on T by setting $f=f_0$ on $T \sim B$, $f=1$ on B. Since $\mu(B)=0$, we have $|\mu_i|(B)=0$ for each i, so $\int f d\mu_i = 0$. The atoms (if any) in $S(f_0)$ are, of course, simply points of T, and the proof is complete.

It is obvious that the first ("if") part of this proof is valid under the (stronger) hypothesis that f be an element of M with $|f(t)| = 1$ for all but at most $n-1$ points; and it would be nice indeed if the existence of such a function were necessary (as well as sufficient) for M to be non-U. This is not the case, however, as the following example will show. (Recall that the space c of all convergent sequences $\{x_n\}$ may be regarded as the space of all continuous functions f on the points $\{1/n\} \cup \{0\}$, with $f(1/n) = x_n$, $f(0) = \lim x_n$.) Our theorem on l_1 says the following about c:

If M has finite codimension n in c, then M is non-U if and only if there exists z in $M^{\perp\perp} \subset l_\infty$ such that $\|z\| = 1$ and $|z_k| < 1$ for at most $n-1$ integers k.

Thus (in particular), if there exists z in M such that $\|z\| = 1$ and $S(z)$ has at most $n-1$ points, then M is non-U. The converse of *this* assertion is false, however. In the example below, let M be the hyperplane $\{x \in c: \sum x_i y_i = 0\}$. There exists z in $(Ry)^\perp = M^{\perp\perp}$ such that $|z_k| = 1$ for all k (so M is non-U), but there is no z in M with this property.

Example. *There exists a sequence y in l_1 such that $(y, z) = \sum y_n z_n = 0$ for some z in l_∞ with $|z_n| = 1$ for all n, but there is no z in c with this property.*

Proof. Let $y_n = (-1)^{n+1} 2^{-n}$, $n=1, 2, 3, \cdots$. If $z_1 = 1$, $z_n = (-1)^n$, $n \geqq 2$, then $\sum y_n z_n = 0$. If $x \in c$, then $(x, y) = y_1 \lim x_n + \sum_{n=2}^{\infty} x_n y_n$. If $|x_n| = 1$ for all n, then there exists n_0 such that $x_k = \lim x_n$ $(= \pm 1)$ for $k \geqq n_0$. This fact, together with some simple computations, shows that $(x, y) = 0$ is impossible for such an element x.

What about the *existence* of subspaces of finite codimension in $C(T)$ which have property U? The question is answered by the following result.

Corollary 3. *If $t_1, t_2, \cdots, t_n$ are distinct points of T, then the subspace $M = \{f \in C(T): f(t_i) = 0, i=1, 2, \cdots, n\}$ has codimension n in $C(T)$ and has property U.*

PROOF. This is immediate from Corollary 2 and the fact that in this case, $M^{\perp}$ is generated by the point masses at $t_1, \cdots, t_n$.

In conclusion, we note that the hypothesis that (T, Σ, μ) be σ-finite was not essential. Indeed, suppose that M is a subspace of $L_1(T, \Sigma, \mu)$ generated by $g_1, g_2, \cdots, g_n$ and that $g_0 \in L_1(T, \Sigma, \mu)$, $g_0 \notin M$. Let T_0 be the union of the supports of the g's i.e., $T_0 = T \sim \bigcap_{k=0}^{n} Z(g_k)$; then T_0 is σ-finite. Furthermore, if we denote by M_0 the subspace of $L_1(T_0, \Sigma, \mu)$ obtained by restricting M to T_0, then g_0 has a unique nearest point in M if and only if the restriction of g_0 to T_0 has a unique nearest point in M_0. Using this remark, it is easy to extend Theorem 1 to the general case.

Added in proof. A simple one-page proof of Liapunov's theorem, using the Krein-Milman theorem, has been given by Joram Lindenstrauss, *A short proof of Liapunoff's convexity theorem*, J. Math. Mech. (to appear).

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