A CHARACTERIZATION OF HEREDITARILY INDECOMPOSABLE CONTINUA

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A continuum (compact, connected Hausdorff space) is said to be indecomposable if it is not the sum of two proper subcontinua. It is said to be hereditarily indecomposable if each of its subcontinua is indecomposable. Knaster [3] showed the existence of such continua in the plane, and Bing [1], [2] showed that there exist hereditarily indecomposable continua of every positive dimension. In the present note we obtain a characterization of such continua by means of certain separation properties.

If M is a continuum and p is a point of M, we recall that the composant of p in M, written $Kp(M)$, is the union of all proper subcontinua of M containing p. Each composant of M is dense in M, and if M is indecomposable then its composants are pairwise disjoint [4]. Further, if the continuum X is not the sum of three continua, no one of which is contained in the sum of the other two, then X is either indecomposable or the sum of two indecomposable continua.

THEOREM. *In order that the continuum S be hereditarily indecomposable, it is necessary and sufficient that for each pair M and N of subcontinua of S, $M-N$ be connected.*

PROOF. If M and N are continua in S and $M-N=A\cup B$, where A and B are separated, then $N\cup A$ and $N\cup B$ are continua, and $N\cup A\cup B$ is a decomposable subcontinuum of S.

If M is a decomposable subcontinuum of S, we may write $M=H\cup K$, where H and K are continua. If $H\cap K$ is connected, then $H\cap K$ is a continuum separating the continuum M. Thus we may assume that C and C' are different components of $H\cap K$. There is a continuum H' contained in H and irreducible from C to C'. Let $a\in H'\cap C$ and $b\in H'\cap C'$. If H' is the sum of the three continua N_1, N_2 and N_3, no one of which is contained in the sum of the other two, we may assume that $a\in N_1$ and $b\in N_2$. If N_1 met N_2, H' would not be irreducible from C to C'. Hence N_3 separates H'. Thus, we need only consider the case where H' is either indecomposable or the sum of two indecomposable continua.

If H' is indecomposable, then $Ka(H')\cap Kb(H')=\emptyset$. Hence, there are points $x\in(H'-K)\cap Ka(H')$ and $y\in(H'-K)\cap Kb(H')$. There

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are subcontinua V and W of H' such that V is irreducible from a to x, W is irreducible from b to y and $V\cap W=\varnothing$, so that $K\cup V\cup W$ is a continuum separated by K.

If H' is not indecomposable, then it is the sum of the two indecomposable continua T and T', say. If T is contained in K, then T' meets C, which is impossible. If T is contained in $K\cup T'$, then $T-T'$ is contained in K. Since $T-T'$ is connected it is contained in C. But then T' meets the closure of $T-T'$ so T' meets C, again a contradiction. We may draw similar conclusions about T'. Thus, there are points $x\in(T-T'\cup K)\cap Ka(T)$, $y\in(T'-T\cup K)\cap Kb(T')$ and continua X and Y of T and T' respectively, such that X is irreducible from a to x, Y is irreducible from b to y and $X\cap Y=\varnothing$, so that K separates the continuum $K\cup X\cup Y$.

Thus, if some subcontinuum of S is decomposable, there exist continua F and F' such that $F-F'$ is not connected, which completes the proof.

References

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