T^*W^*. If, then, $TX \neq W$, there is a sequence w_n^* in W^* such that $\|w_n^*\| \to 1$, while $T^*w_n^* \to 0$. T^* being 1-1 on W^*, it cannot be an open mapping onto T^*W^*, whence the last subspace is not closed.

References

1. S. Banach, *Théorie des operations linéaires*, Monografie Matematyczne, Warsaw, 1932; pp. 145–152.

2. N. Dunford and J. Schwartz, *Linear operators*. I, Interscience, New York, 1958; Lemma 3, p. 488.

University of Illinois

A SHORT PROOF OF JACOBI'S FOUR SQUARE THEOREM

L. CARLITZ[1]

Let $R_4(n)$ denote the number of representations of n as a sum of four squares and let $R_4'(4m)$, where m is odd, denote the number of representations of $4m$ as a sum of four odd squares. It is familiar that

$$R_4'(4m) = 16\sigma(m) \tag{1}$$

and

$$R_4(n) = \begin{cases} 8\sigma'(n) & (n \text{ odd}), \\ 24\sigma'(n) & (n \text{ even}), \end{cases} \tag{2}$$

where

$$\sigma(n) = \sum_{d|n} d, \qquad \sigma'(n) = \sum_{d|n;\, d \text{ odd}} d.$$

These results can be proved rapidly as follows. In the usual notation of elliptic functions put [2, Chapter 21]

$$\lambda = k^2 = \frac{\theta_2^4}{\theta_3^4}, \qquad 1 - \lambda = \frac{\theta_0^4}{\theta^4}\,.$$

Then

Received by the editors May 21, 1965.

[1] Supported in part by NSF grant GP-1593.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use

$$1 - \lambda = \prod_{n=1}^{\infty} \left(\frac{1 - q^{2n-1}}{1 + q^{2n-1}} \right)^8. \tag{3}$$

Now, it follows easily from $q = \exp[-\pi K'/K]$ that [2, p. 521]

$$\frac{1}{q} \frac{dq}{d\lambda} = -\frac{\pi}{K^2}\left(K \frac{dK'}{d\lambda} - K' \frac{dK}{d\lambda} \right)$$
$$= \frac{1}{\lambda(1-\lambda)\theta_3^4}.$$

Thus logarithmic differentiation of (3) yields

$$\theta_2^4 = \lambda\theta_3^4 = 16 \sum_1^{\infty} \frac{(2n-1)q^{2n-1}}{1 - q^{2(2n-1)}} = 16 \sum_{m=1;\, m \text{ odd}}^{\infty} \sigma(m) q^m$$

and (1) follows at once.

Similarly from

$$\lambda = 2q \prod_1^{\infty} \left(\frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8$$

we get

$$\theta_0^4 = (1-\lambda)\theta_3^4 = 1 + 8 \sum_1^{\infty} \left(\frac{2nq^{2n}}{1 + q^{2n}} - \frac{(2n-1)q^{2n-1}}{1 + q^{2n-1}} \right).$$

Replacing q by $-q$ this becomes

$$\theta_3^4 = 1 + 8 \sum_1^{\infty} \left(\frac{2nq^{2n}}{1 + q^{2n}} + \frac{(2n-1)q^{2n-1}}{1 - q^{2n-1}} \right)$$
$$= 1 + 16 \sum_{n=1}^{\infty} \sigma'(n) q^{2n} + 8 \sum_{n=1}^{\infty} \sigma'(n) q(n)$$

and (2) follows at once.

For the standard elliptic function proof of (2) see for example [1, pp. 205–206].

References

1. A. Hurwitz and R. Courant, *Funktionentheorie*, Springer, Berlin, 1929.

2. E. T. Whittaker and G. N. Watson, *Modern analysis*, 4th ed., Cambridge Univ. Press, New York, 1927.

Duke University

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use