$T^*W^*$. If, then, $TX \neq W$, there is a sequence $w_n^*$ in $W^*$ such that $||w_n^*|| \to 1$, while $T^*w_n^* \to 0$. $T^*$ being 1-1 on $W^*$, it cannot be an open mapping onto $T^*W^*$, whence the last subspace is not closed.

References


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**A SHORT PROOF OF JACOBI'S FOUR SQUARE THEOREM**

L. CARLITZ

Let $R_4(n)$ denote the number of representations of $n$ as a sum of four squares and let $R_4'(4m)$, where $m$ is odd, denote the number of representations of $4m$ as a sum of four odd squares. It is familiar that

\[(1) \quad R_4'(4m) = 16\sigma(m)\]

and

\[(2) \quad R_4(n) = \begin{cases} 8\sigma'(n) & (n \text{ odd}), \\ 24\sigma'(n) & (n \text{ even}), \end{cases}\]

where

\[
\sigma(n) = \sum_{d|n} d, \quad \sigma'(n) = \sum_{d|n; d \text{ odd}} d.
\]

These results can be proved rapidly as follows. In the usual notation of elliptic functions put [2, Chapter 21]

\[
\lambda = k^2 = \frac{\theta_2^4}{\theta_3^4}, \quad 1 - \lambda = \frac{\theta_0}{\theta_4^4}.
\]

Then

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(3) \[ 1 - \lambda = \prod_{n=1}^{\infty} \left( \frac{1 - q^{2n-1}}{1 + q^{2n-1}} \right)^8. \]

Now, it follows easily from \( q = \exp\left[ -\pi K'/K \right] \) that [2, p. 521]

\[
\frac{1}{q} \frac{dq}{d\lambda} = -\frac{\pi}{K^2} \left( K \frac{dK'}{d\lambda} - K' \frac{dK}{d\lambda} \right)
= \frac{1}{\lambda(1-\lambda)\theta_3^4}.
\]

Thus logarithmic differentiation of (3) yields

\[
\theta_2^4 = \lambda \theta_3^4 = 16 \sum_{1}^{\infty} \frac{(2n-1)q^{2n-1}}{1 - q^{2(2n-1)}} = 16 \sum_{m=1; m \text{ odd}}^{\infty} \sigma(m)q^m
\]

and (1) follows at once.

Similarly from

\[
\lambda = 2q \prod_{1}^{\infty} \left( \frac{1 + q^{2n}}{1 + q^{2n-1}} \right)^8
\]

we get

\[
\theta_0^4 = (1-\lambda)\theta_3^4 = 1 + 8 \sum_{1}^{\infty} \left( \frac{2nq^{2n}}{1 + q^{2n}} - \frac{(2n-1)q^{2n-1}}{1 - q^{2n-1}} \right).
\]

Replacing \( q \) by \(-q\) this becomes

\[
\theta_3^4 = 1 + 8 \sum_{1}^{\infty} \left( \frac{2nq^{2n}}{1 + q^{2n}} + \frac{(2n-1)q^{2n-1}}{1 - q^{2n-1}} \right)
= 1 + 16 \sum_{n=1}^{\infty} \sigma'(n)q^{2n} + 8 \sum_{n=1}^{\infty} \sigma'(n)q(n)
\]

and (2) follows at once.

For the standard elliptic function proof of (2) see for example [1, pp. 205–206].

**References**


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