UNUSUAL POWER PRODUCTS AND THE IDEAL $[y^2]$

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A power product $P = y_{i(1)}y_{i(2)} \cdots y_{i(n)}$, where $y_{i(j)}$ is the $i(j)$th derivative of y, has *weight* $w = \sum_{j=1}^{n} i(j)$ and *degree* $d = n$. The sequence of integers $(e_1, e_2, \cdots, e_n)$, where $e_k = \sum_{j=1}^{k} i(k) - k(k-1)$, is called the *weight sequence* of P. According to a result of H. Levi ([**1**], Theorem 1.2, p. 545), if some $e_k < 0$, then $P \equiv 0[y^2]$. The discovery by D. G. Mead [**2**] that $Q = y_1y_3y_4y_5 \equiv 0[y^2]$ shows that this criterion is not a necessary condition for membership in $[y^2]$. Q is an *unusual* power product; that is, Q is in $[y^2]$ and has a nonnegative weight sequence. By [**3**], there also exist unusual power products for $[y^p]$, $p > 2$; however, this note is concerned only with $[y^2]$. All unusual power products with weight sequences consisting of 0's, 1's, and 2's are described by Theorem IV, [**2**]:

> A power product P with weight sequence $(e_1, \cdots, e_n)$, $0 \leqq e_i \leqq 2$, $i = 1, \cdots, n$, is unusual if and only if somewhere in the sequence at least one of the following patterns appears: 1, 2, 2, 1; 1, 2, 2, 2, 2, 0; 0, 2, 2, 2, 2, 1; 0, 2, 2, 2, 2, 2, 2, 0.

For particular arrangements of 0's, 1's, 2's, and 3's, similar results may be obtained; for example, using the notation of [**2**],

THEOREM A. *Let* $g(a) = m(1, 3_1, 3_2, \cdots, 3_a, 2, 0)$, $[3_i = 3]$, *then* $g(a) = 0$ *if and only if* $a = 3$.

The purpose of this note is to show that, in general, a finite list of patterns will not suffice to describe unusual power products. By Theorem B there are unusual power products of arbitrarily high degree with no proper factors in $[y^2]$. The following results from [**2**] are stated for easy reference:

(a) $m(A, 0; B, 0) = m(A, 0)m(B, 0)$ for any sequences A and B
(b) $m(1, 0) = -2$
(c) $m(1, 1, C) = -m(1, C)$ for any sequence C
(d) $m(1, 2, 1, D) = 2m(1, D)$ for any sequence D
(e) $m(0, E) = m(E)$ for any sequence E
(f) $m(1, 2, 2, F) = 2m(2, F) + m(1, 2, F)$ for any sequence F.

THEOREM B. *Let* $f(a) = m(1, 2, 3_1, 3_2, \cdots, 3_a, 1)$, $[3_i = 3]$; *then* $f(a) = 0$ *for all* $a \geqq 1$. *Furthermore, if* Q *has the weight sequence* $(1, 2, 3_1, \cdots, 3_a, 1)$, $a \geqq 5$, *then* Q *has no proper factor in* $[y^2]$.

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PROOF. The theorem rcsts on two easy lemmas.

LEMMA I. $m(1, 2, 3, 3, S) = -m(1, 2, 3, S)$, *for any sequence* S.

PROOF. $(1, 2, 3, 3, S)$ is the weight sequence of $Q = y_1y_3y_5y_6 \cdots$. Replacing y_5y_6 and using (a)–(d), (f), we get the equation

$$m(Q) = -2m(3, S) - 2\mathrm{m}(1, 3, S) - 2m(2, 3, S) - m(1, 2, 3, S).$$

Use the equation $m(2, 3, S) = -m(3, S) - m(1, 3, S)$ to complete the proof.

In a similar fashion we prove

LEMMA II. $m(1, 2, 3, 2, S) = 2m(1, 2, S)$, *for any sequence* S.

Returning to the proof of Theorem B, by Lemma I, $f(a) = (-1)^{a-1}m(1, 2, 3, 1)$. But $(1, 2, 3, 1)$ represents the same power product as $(1, 2, 2, 1)$, namely $R = y_1y_3y_4y_5$; and by Theorem IV of [2], $R \equiv 0[y^2]$.

If P is a proper factor of Q, then P is either an α-term or a factor of

$$\begin{aligned} P_1(k) &= y_1y_3y_5y_6y_8 \cdots y_{2k}, \\ P_2(k) &= y_1y_3y_4y_6y_8 \cdots y_{2k}^2, \\ P_3(k) &= y_1y_3y_5^2y_8y_{10} \cdots y_{2k}^2, \\ P_4(k) &= y_0y_3y_5y_6y_8 \cdots y_{2k}^2, \\ P_5(k) &= y_1y_2y_5y_6y_8 \cdots y_{2k}^2. \end{aligned}$$

$P_1(k)$ is comparable to the α-term $y_1y_3y_5$ by Lemma I. $P_2(k)$ has the weight sequence $(1, 2, 2, \cdots, 2, 0)$; hence, is in $[y^2]$ if and only if $k = 4$. By Lemma II, $m(P_3(k)) = 2m(P_2(k-2))$; and by (e) and (c) respectively, $m(P_4(k)) = m(P_2(k-1))$ and $m(P_5(k)) = -m(P_3(k-1))$. Thus Q has no proper factor in $[y^2]$.

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