ELEMENTARY PROOF OF A THEOREM ON CONFORMAL RIGIDITY[1]

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Let $f(z)$ be single valued and analytic in the annulus $A: a<|z|<b$, $f(A)\subset A$, and suppose that f maps every closed curve with winding number $+1$ about $z=0$ onto a curve with the same property. Then [1], [2], [3], [4] f if a rotation. Stimulated by a discussion with Professor A. Marden we shall give a short proof of this result with a minimum of topological notions.

By hypothesis, the integral of

$$\frac{1}{z}-\frac{f'(z)}{f(z)}$$

around any closed curve in A vanishes. Hence there exists a branch, $F(z)$, of $\log\,[z/f(z)]$ single valued in A. Let us define

$$u(re^{i\theta}) = \operatorname{Re} F(re^{i\theta}) = \log r - \log\left| f(re^{i\theta})\right|, \qquad a<r<b,$$

$$I(r) = (2\pi)^{-1}\int_0^{2\pi} u(re^{i\theta})\,d\theta.$$

By Cauchy's theorem applied to $z^{-1}F(z)$, $I(r)$ is independent of r. On the other hand, since $f(A)\subset A$,

$$\log(r/b) \leqq u(re^{i\theta}) \leqq \log(r/a), \qquad a<r<b. \tag{1}$$

Therefore

$$\limsup_{r\to a} I(r) \leqq 0, \qquad \liminf_{r\to b} I(r) \geqq 0.$$

Thus, $I(r)\equiv 0$, $a<r<b$. Hence, by (1),

$$\begin{aligned} J(r) &= (2\pi)^{-1}\int_0^{2\pi}\left| u(re^{i\theta})\right| d\theta = (2\pi)^{-1}\left[\int_{u\geqq 0} u\,d\theta - \int_{u\leqq 0} u\,d\theta\right] \\ &= -\,2(2\pi)^{-1}\int_{u<0} u\,d\theta \leqq 2\log(b/r), \end{aligned} \tag{2}$$

or alternatively,

Received by the editors April 20, 1965.

[1] Work done with support from NSF grant GP 3904 with the University of Minnesota.

$$(3) \qquad J(r) = 2(2\pi)^{-1}\int_{u>0} u\,d\theta \leqq 2\log(r/a), \qquad a < r < b.$$

Since $|u(re^{i\theta})|$ is subharmonic, $J(r)$ is a convex function of $\log r$. Therefore $J(r) \leqq \max[J(r_1),\ J(r_2)]$ whenever $a < r_1 \leqq r \leqq r_2 < b$. Letting $r_1 \to a$, $r_2 \to b$, and using (2), (3) we obtain

$$J(r) \equiv 0, \qquad a < r < b.$$

Thus $u \equiv 0$, $|z^{-1}f(z)| \equiv 1$. Thus $f(z) = e^{ic}z$, c real.

References

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