IDEALS GENERATED BY PRODUCTS

A. P. HILLMAN, D. G. MEAD, K. B. O'KEEFE\(^1\) AND E. S. O'KEEFE

In [1] Levi obtained results on the structure of the differential ideals \([y^n]\) and \([uv]\) and applied these results to the component theory of differential polynomials. The present paper uses Levi's methods to extend his main results on \([uv]\) to \([y_1y_2 \cdots y_n]\). Since the \(y_i\) are independent indeterminates, \([y_1 \cdots y_n]\) is related to \([y^n]\) but is not quite a generalization. The results are motivated by and apply to differential algebra; however, we follow Levi's suggestion at the close of [1] in stating them in the more general form in which \(y_{ij}\) is not necessarily a derivative of \(y_i\).

Let \(y_{ij}\) (\(i=1, \cdots, n\) and \(j=0, 1, \cdots\)) be a set of independent indeterminates over a field \(F\) and let \(R\) be the polynomial ring in the \(y_{ij}\) over \(F\). The signature of a monomial \(M\) in \(R\) is \(D=(d_1, \cdots, d_n)\) if \(M\) has degree \(d_h\) in the \(y_{ij}\) with \(i=h\). The weight of \(M\) is the sum of the \(j\)'s for the factors \(y_{ij}\) of \(M\). A polynomial of \(R\) is homogeneous with signature \(D\) if each of its terms has this signature; it is isobaric of weight \(w\) if each term has weight \(w\).

For \(j=0, 1, \cdots\) let \(x_j\) be a linear combination, with nonzero coefficients in \(F\), of all the products \(y_{1j_1}y_{2j_2} \cdots y_{nj_n}\) of weight \(j\). Let \(I_j\) be the ideal \((x_0, x_1, \cdots, x_i)\) in \(R\). Let \(I=(x_0, x_1, \cdots)\) and let \(Q\) be the quotient ring of \(R\) modulo \(I\). Below we describe functions \(f(D)\) and \(g(D)\) such that a monomial with signature \(D\) and weight \(w\) is in \(I_j\) if \(w<f(D)\) and \(t \geq g(D)\) and such that for every \(D\) and \(w \geq f(D)\) there is a monomial with signature \(D\) and weight \(w\) that is not in \(I\).

2. Levi bases. In the case \(n=2\), Levi obtained the following bases for \(Q\) and \(R\) as vector spaces over \(F\). Let \(u_j = y_{1j}\) and \(v_j = y_{2j}\). A product

\[ P = u_{i_1} \cdots u_{i_r} v_{j_1} \cdots v_{j_s} \]

(1)

of signature \((r, s)\) is an \(\alpha\) term if \(s=0\) or \(j_1 \geq r\) and a \(\beta\) term otherwise. A \(\lambda\) term is of the form \(AX\) with \(A\) an \(\alpha\) term and \(X\) a power product in the \(x_j\). (\(X\) may be 1; thus the \(\lambda\) terms include Levi's \(\alpha\) and \(\gamma\) terms.) Levi showed that the \(\alpha\) terms are a basis for \(Q\) and the \(\lambda\) terms for \(R\).

Let \(P'\) be a factor of \(P\) in (1) selected as follows. If \(P\) is an \(\alpha\) term,

---

Received by the editors April 3, 1965.

\(^1\) The work on this paper by one of the authors was supported by a grant from the American Association of University Women.

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Let $P' = P$. If $P$ is a $\beta$ term, let $e = j_1 + 1$, $S = u_{i_j} v_{i_j}$, and $P' = P/S$. Let $P_i$ be defined by $P_0 = P$ and $P_{i+1} = P_i'$. Let $\theta(P)$ be the expression for $P$ in the form $AS_1 \cdots S_a$ where $a$ is the first $i$ for which $P_i$ is an $\alpha$ term, $A = P_a$, and $S_i = P_i/P_{i-1}$. The weight $h_i$ of $S_i$ satisfies $h_1 \leq h_2 \leq \cdots \leq h_a$. Let $\lambda(P) = A x_{h_1} \cdots x_{h_a}$. Levi showed that $P \rightarrow \lambda(P)$ is a 1-1 mapping of the set of all power products in the $u_j$ and $v_j$ onto the set of all $\lambda$ terms.

3. Extension to general $n$. Let $P = Y_1 Y_2 \cdots Y_n$ where $Y_k$ is a power product in the $y_{ij}$. For $2 \leq k \leq n$ we inductively define $\alpha_k$ terms in the $y_{ij}$, $y_{kj}$ and an expression $\theta_k$ for $Y_1 \cdots Y_k$. When $k = 2$, $\alpha_2$ terms and $\theta_2 = \theta(Y_1 Y_2)$ are defined as in the previous section with $y_{1j}$ in the role of $u_j$ and $y_{2j}$ in that of $v_j$. We assume the definition of $\alpha_m$ terms and of $Y_1 \cdots Y_m$ in the form $\theta_m = A_m S_{m1} \cdots S_{mb}$ where $A_m$ is an $\alpha_m$ term and $S_{mj}$ has signature $(1, \cdots, 1, 0, \cdots, 0)$, with $m$ ones, and weight $h_{mj}$ satisfying $h_{m1} \leq h_{m2} \leq \cdots \leq h_{mb}$. Thinking of $S_{mj}$ as $u_j$ and $y_{m+1,j}$ as $v_j$, let $\theta(S_{m1} \cdots S_{mb} Y_{m+1}) = A^* S_{m+1,1} \cdots S_{m+1,c}$. We then define $\theta_{m+1}$ to be

\[(1)\quad A_{m+1} S_{m+1,1} \cdots S_{m+1,c}\]

where $A_{m+1} = A_m A^*$. The power product $Y_1 \cdots Y_{m+1}$ is defined to be an $\alpha_{m+1}$ term if $c$ in (1) is zero, i.e., if $Y_1 \cdots Y_{m+1} = A_{m+1}$.

As before a $\lambda$ term is of the form $AX$ with $A$ an $\alpha$ (i.e., $\alpha_n$) term and $X$ a power product in the $x_j$. Let $\theta(P) = \theta_n = A S_1 \cdots S_c$ with $A$ an $\alpha$ term and $S_j$ of weight $h_j$ satisfying $h_1 \leq \cdots \leq h_c$ and let $\lambda(P) = A x_{h_1} \cdots x_{h_c}$. The following outline of the process of showing that the $\alpha$ terms are a basis for $Q$ and the $\lambda$ terms for $R$ is essentially as in Levi's work:

Since $x_{h_1}$ is a linear combination of power products one of which is $S_1$, $(P/S_1) x_{h_1} \equiv 0 \pmod{I}$ can be solved as

\[(2)\quad P \equiv f_1 N_1 + \cdots + f_s N_s \pmod{I}\]

where the $f_i$ are in $F$ and the $N_i$ are power products in the $y_{ij}$. Any $N$ in (3) which is not an $\alpha$ term can be replaced by an expression (3) in which $N$ plays the role of $P$. Continuing the process, one ends in a finite number of steps with $P$ congruent to a linear combination of $\alpha$ terms of the same signature and weight as $P$. Thus the $\alpha$ terms generate $Q$. It follows from the process of obtaining the $\theta_k$ and the 1-1 character of $P \rightarrow \lambda(P)$ in the case $n = 2$ that, in the case of general $n$, $P \rightarrow \lambda(P)$ is a 1-1 mapping of the set of power products in the $y_{ij}$ onto the set of $\lambda$ terms. This establishes that the $\lambda$ terms are a basis for $R$. Then the $\alpha$ terms are linearly independent modulo $I$ and form a basis for $Q$. 
The function \( f(D) \) is the minimum of the weights of monomials with signature \( D \) that are not in \( I \); hence \( f(D) \) is the minimum of the weights of the \( \alpha \) terms with signature \( D \). An \( \alpha \) term \( A \) is of the form \( A_{n-1}A^* \) where \( A_{n-1} \) is an \( \alpha_{n-1} \) term and \( A^* \) is an \( \alpha_2 \) term in the \( S_{n-1,i} \) and the \( y_{nj} \). If \( D = (d_1, \ldots, d_n) \) and \( A^* \) is of degree \( t \) in the \( S \)'s, the weight of \( A \) is at least

\[
f(d_1 - t, \ldots, d_{n-1} - t) + td_n
\]

and is \( (4) \) if \( A_{n-1} \) is an \( \alpha_{n-1} \) term of minimal weight for its signature and \( Y_n = (y_{ni})^{d_n} \). Hence \( f(D) \) is the minimum of \( (4) \) for \( 0 \leq t \leq \min (d_1, \ldots, d_{n-1}) \).

There are other ways of defining a basis for \( Q \). Although the basis may be different, the function \( f(D) \) does not depend on the basis. One such alternative shows that

\[
f(D) = \min[f(d_1 - r, \ldots, d_a - r) + f(d_{a+1} - s, \ldots, d_b - s) + \ldots
\]

\[
+ f(d_c - t, \ldots, d_n - t) + f(r, s, \ldots, t)]
\]

where \( r, s, \ldots, t \) range over all nonnegative integers such that the arguments in \( (5) \) are nonnegative.

It will be shown in another paper that \( f(D) \) may be evaluated explicitly as follows. We assume without loss of generality that \( d_1 \leq d_2 \leq \cdots \leq d_n \). For \( 2 \leq i \leq n \) let \( q_i = (d_1 + \cdots + d_i)/(i-1) \) and let \( k \) be the smallest \( i \) for which \( q_i \) assumes its minimum. Let \( q \) and \( r \) be integers defined by \( d_1 + \cdots + d_k = (k-1)q + r \) and \( 0 \leq r < k-1 \). Let \( c_i = q - d_i \) for \( i = 1, \ldots, k \). Let \( \sigma_1 = c_1 + \cdots + c_k \) and \( \sigma_2 = \sum_{i<j} c_i c_j \). Then \( f(D) = \sigma_2 + r\sigma_1 + [(r+1)r/2] \).

It is easily seen from Levi's process that \( g(d_1, d_2) \) may be chosen as \( d_1 + d_2 - 2 \). This and the process described above show that \( g(d_1, \ldots, d_n) \) may be chosen as

\[
d_n + \min(d_1, \ldots, d_{n-1}) - 2.
\]

Using the symmetry of the ideal \( I_i \) in the subscripts \( i \) of the \( y_{ij} \), this may be improved to \( g(D) = \min_{i \neq j} (d_i + d_j) - 2 \).

References
