THE DIFFERENTIAL IDEAL $[uv]$

KATHLEEN B. O'KEEFE AND EDWARD S. O'KEEFE

1. Introduction. Let $R\{u, v\}$ be the polynomial ring $R[u, u_1, u_2, \cdots; v_1, v_{l+1}, \cdots]$ over $R$, a field of characteristic zero, with the derivation $D(y_i) = y_{i+1}$ for $y = u$ or $v$.

Let $\Omega = [uv]$ be the differential ideal generated by the form $X = uv_l$. $\Omega$ has the same elements as the ideal $(uv_l, (uv_l)_1, (uv_l)_2, \cdots)$, where the subscripts again denote derivatives.

A power product in $R\{u, v\}$ $P = u_{i(1)}u_{i(2)} \cdots u_{i(m)}v_{j(1)}v_{j(2)} \cdots v_{j(n)}$ is of weight, $w(P) = \sum_{k-1}^{m} i(K) + \sum_{l-1}^{n} j(P)$, and signature, $\text{sig}(P) = (m, n)$.

The following fundamental theorem is proved in [3].

**Levi's Theorem.** If $P$ is a power product in $R\{u, v\}$ and $w(P) < m \cdot n$, then $P$ is in the ideal $[uv]$.

The purpose of this paper is to show that if $P$ contains no proper factor which is in $[uv]$, and if $w(P) \geq mn$, then $P$ is not in $[uv]$.

2. Derivations and isomorphic images of $R\{u, v\}$. Computations in $R\{u, v\}$ are simplified by working in an isomorphic image of $R\{u, v\}$, $R\{\bar{u}, \bar{v}\}$. $R\{\bar{u}, \bar{v}\}$ is the ring $R[\bar{u}, \bar{u}_1 \cdots, \bar{v}, \bar{v}_1 \cdots]$ with derivation $\bar{D}(\bar{y}_i) = \bar{y}_{i+1}$ for $y = u$ or $v$. The isomorphism is established by the mapping $h: u_i \mapsto u_i/i!$, $v_j \mapsto v_j/j!$. Thus $\bar{D}(\bar{u}_i)$ corresponds to $D(u_i)/(i+1)$ and $\bar{D}(\bar{v}_j)$ to $D(v_j)/j+1$. For typographical convenience, the bars will be omitted; hence $\bar{D}^n(\bar{u}\bar{v})$ is written $(uv)_n = \sum_{j=0}^{n} u_j v_{n-j}$.

**Definition 2.1.** $\bar{D}_i = D_i$ is defined on $R[u, u_1, \cdots, v_1, v_{l+n}, \cdots]$ by

1. $D_i(u_j) = (i + 1)u_{i+1}$ for $i \geq 0$.
2. $D_j(v_i) = \begin{cases} (j - l + 1)v_{j+1} & \text{for } j \geq l, \\ 0 & \text{for } j < l. \end{cases}$
3. If $D_k$ has been defined, then $D_{k+1} = D(D_k)$.

**Theorem 2.2.** Let $h$ be the (nondifferential) isomorphism of $R = R[\bar{u}, \bar{u}_1, \cdots, v, v_1, \cdots]$ onto $R_1 = R[u, u_1, \cdots, v, v_{l+1}, \cdots]$ determined by mapping $u_i$ into $u_i$ and $v_i$ into $v_{i+1}$. Then

\[
(2.1) \quad h(D^0(P)) = D^1(h(P)).
\]
Proof. It suffices to show (2.1) for $P = u_i$ and $P = v_j$. Suppose that $i \geq 0$, then for $l \geq 0$, $h(D^0(u_i)) = h((i + 1)u_{i+1}) = (i+1)u_{i+1} = D^1(h(u_i))$; and for $j \geq l$, $h(D^0(v_j)) = h((j+1)v_{j+1}) = (j+1)v_{j+1} = D^1(v_{j+1}) = D^1(h(v_j))$.

Corollary 2.3. $\mathfrak{R}_1$ is closed under the operation $D^1$. Furthermore, the ideal $[uv_1]$, the image of $[uv]$ under the mapping $h$, is closed under $D^1$.

Corollary 2.4. Let $R\{u, v_1\}$ be the Ritt algebra $\langle \mathfrak{R}_1, D^1 \rangle$, then $R\{u, v_1\}$ is isomorphic to $R\{u, v\}$.

Let $(uv)$ be the (algebraic) subring of $R\{u, v\}$ generated by $uv$; that is, $(uv)$ is the set of elements of $R\{u, v\}$ divisible by $uv$.

Theorem 2.5. There is a module isomorphism $g$ which maps $uR\{u, v\}/(uv)$ onto $R\{u, v_1\}$.

Proof. Let $I = (uv)$. If $a \in uR\{u, v\}/(uv)$, then for a unique $b$ not involving $v$, $a = ub + I$. Define $g$ by $g(a) = b$. Then $g(I) = 0$, $g(c + I) = c/u$, if $c$ does not contain $v$. Clearly if $r \in R$, $g(ra) = rg(a)$ and if $a_1$ and $a_2$ are elements of $uR\{u, v\}/(uv)$, then $g(a_1 + a_2) = g(a_1) + g(a_2)$.

Furthermore, for every $c$ in $R\{u, v_1\}$, $g^{-1}(c) = uc + I$ and $g^{-1}(c)$ is an element of $uR\{u, v\}/(uv)$.

Theorem 2.6. Under $g$, $u[uv]/(uv)$ in $R\{u, v\}$ is mapped isomorphically on $[uv_1]$ in $R\{u, v_1\}$.

Proof. If $a \in u[uv]/(uv)$, then $a = uc + I$, where $c = \sum_{i=0}^{n} d(i)(uv)_i$ with $d(i) \in R\{u, v\}$. For $i > 0$, $(uv)_i = (uv_1)_{i-1} + u_iv$; hence, $uc + I = u \sum_{i=0}^{n} d(i)(uv_1)_{i-1} + I$. Thus $g(a) = c$ and $c \in [uv_1]$. Further, $g^{-1}g(a) = a$. If any $c$ is in $[uv_1]$, then $g^{-1}(c) = uc + I$, or $u \sum_{i=0}^{n} d(1)(uv_1)_i + I$.

But then certain elements of $I$ may be used to fill out the sums because $ud(i)u_iv \in I$ for every $i$. Therefore $u \sum_{i=0}^{n} d(i)(uv_1)_i + I = u \sum_{i=0}^{n+1} d(i-1)(uv)_i + I$, and $g$ covers all of $[uv_1]$ and is an isomorphism.

Corollary 2.7. If $Q \equiv 0[uv_1]$, then $u \cdot Q \equiv 0 [uv]$.

Proof. Using the $g^{-1}$ of Theorem 2.2, $[uv_1]$ is mapped onto $u[uv]/(uv)$. Hence $uQ \equiv 0[uv]$ because $uQ \equiv uQ + I = g^{-1}(Q)$.

3. The operator $T_n$. Let $P = u_jUV$ be a power product of signature $\langle k, l \rangle$ and excess weight zero.

Definition 3.1. $T_n$ operates on $V$ and is defined by
1. For $n = 1$, $T_1(V) = D^1(V) - D^0(V)$.
2. If $T_{n-1}(V)$ has been defined, then $T_n(V) = D^1(T_{n-1}(V)) - T_{n-1}(D^0(V))$. (Note that $T_n$ and $D^1$ do not commute.)
Theorem 3.2. Let \( V = v_{j(1)}v_{j(2)} \cdots v_{j(l)} \), then for \( n \leq l \), \( T_n(V) = (-1)^n n! \sum v_{j(1)} \cdots v_{i_t+1} \cdots v_{i_{(n)+1}} \cdots v_{j(l)} \), with the summation extending over all products in which exactly \( n \) \( v \)-subscripts are raised by 1. (That is, no \( j(i), i = 1, \ldots, l \), is raised more than 1.) If \( n > l \), \( T_n(V) = 0 \).

Proof. The proof is by induction on \( n \), keeping \( l \) fixed.

For \( n = 1 \),
\[
T_1(V) = D^1(V) - D^0(V) = \sum_{m=1}^l (j(m))v_{j(1)} \cdots v_{j(m)+1} \cdots v_{j(l)} - \sum_{m=1}^l (j(m) + 1)v_{j(1)} \cdots v_{j(m)+1} \cdots v_{j(l)} = -\sum_{m=1}^l v_{j(1)} \cdots v_{j(m)+1} \cdots v_{j(l)}.
\]

For \( n > 1 \), assume that the theorem holds for values less than \( n \). Let \( Z_n \) be the set of all functions \( z \) on \( \{1, 2, \ldots, l\} \) to \( \{0, 1\} \) with \( n \) occurrences of 1. The induction hypothesis may now be written, for \( p < n \),
\[
T_p(V) = (-1)^p p! \sum_{z \in Z_p} v_{j(1)+z(1)} \cdots v_{j(l)+z(l)}.
\]

By definition \( T_n(V) = D^1(T_{n-1}(V)) - T_{n-1}(D^0(V)) \), and the induction hypothesis may be applied to \( T_{n-1} \). Using the definition of \( D^0 \) and \( D^1 \), an expression for \( T_n \) may be derived as follows.

\[
T_n(V) = D^1\left((-1)^{n-1}(n - 1)! \sum_{z \in Z_{(n-1)}} v_{j(1)+z(1)} \cdots v_{j(l)+z(l)}\right) - T_{n-1}\left(\sum_{l=1}^i (j(l) + 1)v_{j(1)} \cdots v_{j(l)+1} \cdots v_{j(l)}\right)
= (-1)^{n-1}(n - 1)!
\]

\[
\cdot \sum_{z \in Z_{(n-1)}} \left(\sum_{l=1}^i (j(l) + z(l))v_{j(1)+z(1)} \cdots v_{j(l)+z(l)+1} \cdots v_{j(l)+z(l)}\right)
- \sum_{l=1}^i (j(l) + 1)(-1)^{n-1}(n - 1)!
\cdot \left(\sum_{z \in Z_{(n-1)}} v_{j(1)+z(1)} \cdots v_{j(l)+1+z(l)} \cdots v_{j(l)+z(l)}\right).
\]
These two sums are exactly comparable, the same $t$'s and $z$'s occurring in each. The sign of one term is $+$ and the other $-$; the sum of the coefficients being

$$j(t) + z(t) - (j(t) + 1).$$

The sum coefficient is then $-1$ for exactly those terms where $z(t) = 0$. It is $0$ for the others. For $n \leq l$, then, the terms unify, giving for each a new $z$, an element of $Z_n$; and for $n > l$, the terms cancel. In case $n \leq l$, each element of $Z_n$ can be found in $n$ ways from as many elements of $Z_{n-1}$; hence, the new factor in the coefficient is $-n$. This concludes the proof.

The $T$-operator will now be applied to an arbitrary power product $P$ of excess weight zero. First of all, if $P$ contains any factor of negative excess weight, then $P$ is in $[u v]$. Therefore, in particular, assume that $P$ does not contain $u v$.

**Theorem 3.3.** Let $P = u_1 U V$, then $P \equiv u U T_1(V) [u v]$.

**Proof.** Since $P$ is a power product of excess weight zero, $u U V$ has negative excess weight and is zero modulo $[u v]$ by Levi's Theorem. Mapping $\mathfrak{R}$ into itself by $D^0$, $u U V = 0 [u v]$ becomes

$$u_1 U V + u D^0(U)V + u U D^0(V) \equiv 0 [u v].$$

Consider $Q = U V$ as a power product in $\mathfrak{R}_1$. Then $S = U h^{-1}(V)$ in $\mathfrak{R}$ has signature $(k - 1, l)$ and weight $w = kl - 1 - l < (k - 1)l$; hence $S \equiv 0 [u v]$. Under $D^0$, $S \equiv 0 [u v]$ becomes

$$D^0(U)(h^{-1}(V)) + U D^0(h^{-1}(V)) \equiv 0 [u v].$$

Mapping $\mathfrak{R}$ into $\mathfrak{R}_1$, (3.2) becomes

$$D^0(U)V + U D^1(V) \equiv 0 [u v_1].$$

The derivation of $R\{u, v_1\}$, $D'$, may be used in $[u v]$ because using the mapping $g$ of Theorem 3.5, $g^{-1}D^g$ maps $u R\{u, v\} / (u v)$ into itself and $u [u v] / (u v)$ into itself. Hence, by Corollary 2.7,

$$u D^0(U)V + u U D^1(V) \equiv 0 [u v].$$

Substituting (3.4) in (3.1) completes the proof.

**Lemma 3.4.** Let $P = u_1 U V$ and let $h$ map $\mathfrak{R}$ onto $\mathfrak{R}_1$. If $Q = U h^{-1}(T_{j-1}(V))$, then $Q \equiv 0 [u v]$.

**Proof.** By Theorem 3.2, $w(T_{j-1}(V^*)) = w(V) + (j - 1)$ for each term $T_{j-1}(V^*)$ in $T_{j-1}(V)$. For each term $Q^*$ in $Q$, $w(Q^*) = w(P) - j$
\(+(j-1)-l=kl-l-1<(k-1)l\); and the signature of \(Q^*\) is \((k-1, l)\). Hence \(Q^*\equiv 0[wv]\) by Levi's Theorem.

**Theorem 3.5.** Let \(P = u_j UV\), then for all \(j > 0\),

\[
P \equiv \frac{1}{j!} uUT_j(V) [wv].
\]

**Proof.** The proof is by induction on \(j\), and the case \(j = 1\) is covered by Theorem 3.3. Assume that (3.5) holds for values less than \(j\). In \(R\), \(u_{j-1} UV \equiv 0[wv]\) by Levi's Theorem. Under \(D^0\), we have

\[
j u_j UV \equiv (-u_{j-1} D^0(U)V - u_{j-1} U D^0(V)) [wv].
\]

Applying the induction hypothesis to each term on the right (3.6) becomes

\[
j u_j UV \equiv \left( \frac{1}{(j - 1)!} uD^0(U)T_{j-1}(V) \right.
\]

\[
- \left. \frac{1}{(j - 1)!} uUT_{j-1}(D^0(V)) \right) [wv].
\]

Map \(R\) onto \(R_1\) by \(h\) and consider \(Q = UT_{j-1}(V)\) as a power product in \(R_1\). Then \(S = Uh^{-1}(T_{j-1}(V))\) is in \([wv]\) by Lemma 3.4. Under \(D^0\), \(S \equiv 0[wv]\) becomes

\[
D^0(U)h^{-1}(T_{j-1}(V)) + UD^0(h^{-1}(T_{j-1}(V))) \equiv 0 [wv].
\]

Mapping \(R\) onto \(R_1\), (3.8) becomes

\[
D^0(U)T_{j-1}(V) + UD^1(T_{j-1}(V)) \equiv 0 [wv_1].
\]

By Corollary 2.7, we get

\[
u D^0(U)T_{j-1}(V) + uUD^1(T_{j-1}(V)) \equiv 0 [wv].
\]

Substituting (3.10) in (3.7) completes the proof.

4. The converse of H. Levi's Theorem for \([wv]\). Let \(P = u_{i(1)}u_{i(2)}\ldots u_{i(k)}v_{j(1)}v_{j(2)}\ldots v_{j(l)}\) be of signature \((k, l)\) and weight \(w\). Assume that \(P\) has no factor of negative excess weight. By Theorem III of [4], without loss of generality, we may set \(w(P) = kl\). If a sequence of \(k\) transformations exist such that

\[
\begin{align*}
(1) \ V &= v_{j(1)}\ldots v_{j(l)} \text{ is changed to } v^t_k, \\
(2) \text{ in the } t\text{th transformation exactly } i(t) \ v\text{-subscripts are increased by one,} \\
(3) \ U &= u_{i(1)}\ldots u_{i(k)} \text{ is changed to } u^k;
\end{align*}
\]

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
then $P$ may be written congruent to a linear combination of $\alpha$-terms of the same weight and signature as $P$, [3]. $P$ is of excess weight zero and thus $P \equiv c u^k v_i^j [uv]$. The only question concerns the coefficient $c$, which is not zero, but is $(-1)^{i_1+i_2+\cdots+i_m}$ where $m$ is the number of sequences which transform $V$ to $v_i^j$. Thus $c = 0$ if and only if $m = 0$, and we have proved

**Theorem 4.1.** If $P = UV$ has a nonnegative weight matrix, then $P$ is not in $[uv]$ if and only if $V$ can be transformed to $v_i^j$ by a sequence of $n$ steps, in the $t$th of which exactly $i(t)$ $v$-subscripts are increased by one.

It remains to characterize those $U$ and $V$ for which (4.1) exists.

At the $t$th step, suppose a power product $M$ is transformed into a power product $N$ as follows: $u_i$ in $M$ is replaced with $u$ and the lowest $j(t)$ $v$-subscripts (assuming that $j(1) \leq j(2) \leq \cdots \leq j(t)$) are increased by one. Now, if $N$ contains a factor with negative excess weight, then the same is true of $M$. More generally, we prove

**Theorem 4.2.** Let $M$ be a power product of signature $\langle k, l \rangle$ containing $u_i, t > 0$ and $t$ $v$'s, $v_{j(1)} \cdots v_{j(t)}$, and let

$$N = M \frac{u v_{j(1)} + \cdots + v_{j(t)} + 1}{u v_{j(1)} \cdots v_{j(t)}}.$$

Then if $G$ is any factor on $N$ with excess weight $e(G)$, there is a factor $F$ of $M$ with excess weight $e(F) \leq e(G)$.

**Proof.** We may assume $G$ has $u$ as a factor; otherwise, by reducing the subscripts in $G$ that have been raised we get a factor $G^*$ of $M$ with $e(G^*) \leq e(G)$. Therefore $G$ is of the form $u U_1 V$, where $U_1$ is a factor of $U$; notationally, let $U_1 = U$. If $V$ involves no unchanged subscripts, then lowering the $n$ subscripts of $V$ that have been raised we get $V^*$ and a factor $U V^*$ of $M$ with $e(U V^*) = w(U) + w(V) - n - (k-1)n = e(u U V)$. If $V$ involves all the changed subscripts, then similarly $e(u_i U V^*) = t + w(U) + (W(V) - t) - k \deg V^* = e(u U V)$. If $V$ involves an unchanged subscript but not all changed ones, we can exchange an unchanged subscript for a changed one except in the case that all the changed subscripts of $N$ are $j(t) + 1$ and all the unchanged subscripts of $G$ are $j(t)$. Thus a reduction is achieved except in the case that $G$ is of the form $u U v^p_{j(t) + 1} v^q_{j(t)}$, $p < t, q > 0$. Consider the cases (1) $k \geq j(t) + 1$ and (2) $k \leq j(t)$.

In case 1,

$$e(u U v^p_{j(t) + 1} v^q_{j(t)}) \leq e(u U v^p_{j(t)} + v^q_{j(t)}).$$
In case 2,

$$e(uU_{v_j(t)+1}^q) \leq e(uU_{v_j(t)+1}^p).$$

In either case, a factor F of M has been found such that $e(F) \leq e(G)$, and the proof is complete.

**Corollary 4.3.** If $P = UV$ has a nonnegative weight matrix and excess weight zero, then $P \neq 0[uv]$.

**Proof.** By symmetry we may assume that $V \neq 0(v)$. By Theorem 4.2, there is a sequence of transformations satisfying (4.1) which transforms $P$ into the $\alpha$-term $u^k v^l$.

**Corollary 4.4.** If $P = u^i v_j$, the smallest exponent $q$ such that $P^q \equiv 0[uv]$ is $q = i + j + 1$.

**Proof.** $Q = (u,v_j)^{i+j+1}$ has negative excess weight; hence, by Levi's Theorem is in $[uv]$. On the other hand, $S = (u,v_j)^{i+j}$ has a nonnegative weight matrix, excess weight zero, and is not in $[uv]$ by Corollary 4.3. This solves Ritt's exponent problem for $[uv]$, ([1], p. 177).

**Bibliography**