GENERALIZATION OF DEHN'S RESULT ON THE CONJUGACY PROBLEM

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1. **Introduction.** In 1911 Max Dehn [2] solved the conjugacy problem for the fundamental group G_k of a two-dimensional orientable closed manifold of genus k. Now G_k can be finitely presented in terms of $2k$ generators and one defining relation as follows:

$$G_k = gp(a_1, b_1, a_2, b_2, \cdots, a_k, b_k; a_1^{-1}b_1^{-1}a_1b_1 \cdots a_k^{-1}b_k^{-1}a_kb_k = 1).$$

If $k > 1$ then G_k is the free product of two free groups with an infinite cyclic group amalgamated. Hence the following theorem generalizes Dehn's result.

THEOREM. *Let G be the free product of free groups F_λ with an infinite cyclic group amalgamated. Then G has a solvable conjugacy problem.*

We remark that the word problem has already been solved for G (e.g. Lipschutz [4]) and that Greendlinger [3] extended Dehn's result to a class of groups which overlap those of our theorem.

A theorem of Solitar [5] which gives conditions for the solution of the conjugacy problem for free products with amalgamations is used in the proof of our theorem.

2. **Preliminary results.** Let F be a free group. We will let u, v and w denote freely reduced nonempty words in F; the letters $a, b, c, \cdots$ will denote a fixed set of generators of F and hence the letters of the words in F. We use the notation

$l(u)$ for the length of u;
$v \equiv u_1u_2$ if v is identical to the letters of u_1 followed by the letters of u_2.

Recall that $u = a_1a_2 \cdots a_n$ is cyclically reduced iff $a_n \neq a_1^{-1}$.

LEMMA 1. *Suppose u is cyclically reduced and $v \not\equiv u^{-1}w$. If the first letter in the product*

$$uuvu^n \tag{1}$$

is absorbed when (1) is free reduced, then $n < 0$ and u and v commute.

PROOF. We will perform cancellations in (1) from left to right.

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Since u is cyclically reduced, cancellations can only take place in two places, on either side of v. Let u_1 denote the first u in (1) and let $u_1=a_1a_2\cdots a_m$. Then (1) can be written

$$a_1a_2\cdots a_muvu^n.$$

Now a_1 can be absorbed only if a_m is absorbed. Since $v\not\equiv u^{-1}w$, a_m cannot be absorbed by a letter of v and, furthermore, a_m can be absorbed only after all the letters of v are absorbed. We now consider the step in the free reduction of (1) just before a_m is absorbed. If $n\geqq 0$ and a_m is absorbed by the letter a_i of $u_2=u=a_1a_2\cdots a_m$, then (1) must be reduced to the form

$$a_1a_2\cdots a_na_ia_{i+1}\cdots a_nu^r.$$

Thus if a_1 is absorbed then $(a_ia_{i+1}\cdots a_n)^2=1$. But there are no elements of finite order in a free group; hence if a_1 is absorbed, then $n<0$ as claimed.

Now, for $n<0$, let a_m be absorbed by the letter a_i^{-1} of $u_2^{-1}=u^{-1}$ $=a_m^{-1}a_{m-1}^{-1}\cdots a_1^{-1}$. Then (1) is reduced to the form

$$a_1a_2\cdots a_ma_i^{-1}a_{i-1}^{-1}\cdots a_1^{-1}u^r,\qquad r<0.$$

If $i=m$, then $uvu^{n-r-1}=1$ which implies that v if a power of u and so commutes with u. On the other hand, if $i<m$ and a_1 is absorbed, then

$$(a_1a_2\cdots a_i)(a_{i+1}\cdots a_m)(a_i^{-1}\cdots a_1^{-1})(a_n^{-1}\cdots a_{i+1}^{-1}) = 1.$$

Hence $a_1a_2\cdots a_i$ and $a_{i+1}\cdots a_m$ commute and so are powers of the same element, say w. Thus u is a power of w. Furthermore

$$uvu^ta_m^{-1}\cdots a_{i+1}^{-1} = 1$$

and so v is also a power of w. Accordingly, u and v commute.

COROLLARY 1. *Let u be cyclically reduced and let $m>l(v)+2$. Then either*

$$m < l(u^mvu^n) + l(v) + 2$$

or u and v commute.

PROOF. We can write v in the form $v\equiv u^{-s}v^*$ where $0\leqq s\leqq l(v)$ and $v^*\not\equiv u^{-1}w$. Then $u^mvu^n=u^{m-s}v^*u^n$. Let u_1 denote the second u preceding v^*. Suppose the first letter of u_1 is not absorbed when u^mvu^n is freely reduced; then

$$l(u^mvu^n) = l(u^{m-s}v^*u^n) > m - s - 2 \geqq m - l(v) - 2.$$

On the other hand, if the first letter of u_1 is absorbed then, by Lemma 1, u and v^* commute and so are powers of some element w. Hence v is also a power of w and so u and v commute as claimed.

REMARK. This corollary extends a lemma of Baumslag [**1**, p. 428].

LEMMA 2. *Given u and v, one can decide whether or not u is a power of v or is conjugate to a power of v; and such a power is unique.*

PROOF. If $u = w^{-1}u^m w = (w^{-1}uw)^m$, then

$$|m| \leqq l(u) = l((w^{-1}uw)^m).$$

Thus we only need to test a finite number of cases. The lemma now follows from the fact that the conjugacy problem is solved for free groups, and the fact that different powers of an element in a free group are in different conjugate classes.

3. Main results.

LEMMA 3. *Let G be the free product of free groups F_λ with an infinite cyclic group amalgamated. Let h be a generator of the amalgamated cyclic group and suppose h is cyclically reduced in each factor. Let u and v be elements of G with free product length n, say $u = u_1u_2 \cdots u_n$ and $v = v_1v_2 \cdots v_n$ where u_i and v_j appear in individual factors. Then we can decide if there exists an m such that*

$$h^m u_1 u_2 \cdots u_n h^{-m} = v_1 v_2 \cdots v_n. \tag{2}$$

PROOF. Since the word problem has been solved for G, we can decide if h and u_1 commute. Suppose h and u_1 do not commute. By properties of free products with amalgamations, if (2) holds then $v_1^{-1}h^m u_1$ belongs to the amalgamated cyclic group; say $v_1^{-1}h^m u_1 = h^{-r}$. Then

$$h^m u_1 h^r = v_1.$$

Hence, by Corollary 1, $|m| < l(v_1) + l(u_1) + 2$. The lemma now follows, in the case that h and u_1 do not commute, from the fact that the word problem has been solved for G and that there are only a finite number of m to consider.

Suppose h and u_1 do commute. Then $v_1^{-1}u_1$ belongs to the amalgamated cyclic group; say $v_1^{-1}u_1 = h^s$ and so $u_1 = v_1 h^s$. (By Lemma 2, s is unique and can be found.) Substituting in (2) and using the fact that u_1 and h commute gives

$$v_1 h^m (h^s u_2) u_3 \cdots u_n h^{-m} = v_1 v_2 \cdots v_n$$

or

$$h^m(h^s u_2)u_3 \cdots u_n h^{-m} = v_2 v_3 \cdots v_n.$$

The lemma now follows from induction.

PROOF OF THEOREM. Let h be a generator of the amalgamated cyclic group. The theorem is proven if we can decide whether or not any two free product cyclically reduced elements u and v in G are conjugate. We can also assume without loss in generality that h is cyclically reduced in the factors in which u and v appear (by considering appropriate generating sets of these factors).

Case I. u and v have different free product lengths. Then u and v are not conjugate.

Case II. u and v have length 1; say $u \in F_i$ and $v \in F_j$. By Solitar's theorem, either (a) $i=j$ and u and v are conjugate elements of F_i; or (b) there exist a sequence of elements $h_1, h_2, \cdots, h_s$ in the amalgamated subgroup such that u and h_1 are conjugate elements of F_i, h_1 and h_2 are conjugate elements of F_{λ_1}, h_2 and h_3 are conjugate elements of $F_{\lambda_2}, \cdots, h_s$ and v are conjugate elements of F_j. But the elements of a cyclic subgroup of any factor are in different conjugate classes; hence $h_1 = h_2 = \cdots = h_s = h^m$ for some integer m.

Now in case (a) we can decide if u and v are conjugate elements of F_i since the conjugacy problem has been solved for free groups. In case (b) we can decide, by Lemma 2, if u is conjugate to a power h^m of h in F_i and if v is conjugate to the same power h^m in F_j. Consequently, in Case II, we can decide if u and v are conjugate elements in G.

Case III. u and v have length $n>1$; say $u = u_1 \cdots u_n$ and $v = v_1 \cdots v_n$ where u_i and v_j appear in factors. By Solitar's theorem, u and v are conjugate iff for some i and some j,

$$h^m u_i \cdots u_n u_1 \cdots u_{i-1} h^{-m} = v_j \cdots v_n v_1 \cdots v_{j-1}.$$

By Lemma 3, we can decide if the above is true. Since there are only n^2 cases to consider, we can decide if u and v are conjugate elements of G.

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