COUNTABLE DIRECT SUMS OF TORSION COMPLETE GROUPS

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1. **Introduction.** All groups considered in the sequel are abelian p-groups without elements of infinite height. Topological notions refer to the p-adic topology. Notation and terminology follow [2], the main exception being that groups in which every bounded Cauchy sequence converges are called torsion complete, the word "closed" being reserved for its topological sense.

In [4] Kolettis introduced the notion of a semicomplete group, i.e. a group which is the direct sum of a torsion complete group and a direct sum of cyclics, and showed that any two such decompositions had isomorphic refinements. These groups form a natural subject for investigation, subsuming the two classes of most understood groups. The more extensive class, playing the title role here, has the same attraction and is considerably less ad hoc. The main objective of this paper is to prove the isomorphic refinement theorem for any two decompositions of a group in this class.

2. **Basic weapons.** We begin with a lemma concerning extending purifications of subgroups of the socle of a torsion complete group.

LEMMA 1. *Let G be a torsion complete group, H a subgroup of $G[p]$, K a pure subgroup of G, $K[p]\subseteq H$. There exists a pure subgroup L of G such that $K\subseteq L$ and $L[p]=H$.*

PROOF. By Zorn's lemma pick L maximal among the pure subgroups of G containing K whose socles are contained in H. Suppose $x\in H-L$. Since G is torsion complete, the reduced part of G/L is torsion complete. Therefore $x+L$ is contained in a rank-one summand S of G/L. The inverse image S_0 of S is pure in G, contains L and x, and $S_0[p]\subseteq H$, contradicting the maximality of L.

The next lemma reduces the notion of torsion completeness to the socle.

LEMMA 2. *Let G be a p-group without elements of infinite height. Then G is torsion complete if $G[p]$ is complete* (*in the G-topology*).

PROOF. We show by induction that $G[p^n]$ is complete. Let $\{x_j\}$ be a Cauchy sequence in $G[p^n]$. Then $\{px_j\}$ is a Cauchy sequence in

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$G[p^{n-1}]$. By induction, $\{px_j\}$ converges to some element $y=pz$ (pG is closed). Therefore $\{p(x_j-z)\}$ converges to 0 and we can find elements $g_r\in G$ and a subsequence $\{x_{j(r)}\}$ such that $p(x_{j(r)}-z)=p^r g_r$. Now $\{x_{j(r)}-z-p^{r-1}g_r\}$ is a Cauchy sequence in $G[p]$ and so converges to $h\in G[p]$. Therefore $\{x_{j(r)}\}$ and hence $\{x_j\}$ converges to $h+z$.

Combining these results we show our class to be closed under summands.

THEOREM 1. *Let $G=\sum_1^\infty G_j$, G_j torsion complete. Suppose H is a summand of G. Then $H=\sum_1^\infty H_j$, H_j torsion complete.*

PROOF. By Lemma 1 we can define a chain of pure subgroups $\{T_i\}$ satisfying $T_i\subseteq\sum_1^i G_j$ and $T_i[p]=H[p]\cap\sum_1^i G_j$. But $T_i[p]$ is closed in G, being the intersection of two closed subgroups, and hence in $\sum_1^i G_j$. Since $\sum_1^i G_j$ is torsion complete and T_i is pure, $T_i[p]$ is complete in T_i and so, by Lemma 2, T_i is torsion complete. Let $\prod_H$ be the projection of G onto H. Then $\prod_H T_i\cong T_i$ is torsion complete and pure in H (socle elements have correct heights) and therefore is a summand of H. We thus have a chain of torsion complete summands of H, $\prod_H T_1\subseteq\prod_H T_2\subseteq\cdots$ such that $\bigcup\prod_H T_i=H$ (since it is pure and contains $H[p]$). Define H_j such that $\prod_H T_{j-1}\oplus H_j=\prod_H T_j$.

3. **Isomorphic refinements.** We hasten to the key lemma for the isomorphic refinement theorem, showing that a torsion complete summand of a direct sum of torsion complete groups can be pulled out piece by piece. First a technical matter.

LEMMA 3. *Let $G=\sum_J A_j$, J an arbitrary index set. If H is a torsion complete summand of G then there exists a finite subset I of J such that $H[p]-\sum_I A_j$ has bounded height.*

PROOF. Suppose not. Select finite subsets $I_1\subset I_2\subset\cdots$ of J and elements $x_n\in H[p]-\sum_{I_n}A_j$ such that $x_n\in H[p]\cap\sum_{I_{n+1}}A_j$ and x_n has height exceeding p^n. By virtue of Lemma 1 there exists a chain $\{H_n\}$ of pure subgroups of H satisfying $H_n[p]=H[p]\cap\sum_{I_n}A_j$. As in the proof of Theorem 1, each H_n is torsion complete. By the same token $\bigcup H_n$ is torsion complete. Thus $H_{n+1}=H_n\oplus H_n'$ and H_n' has a nonzero element of height exceeding p^n, namely $\prod H_n'(x_n)$. But $\bigcup H_n=H_1\oplus H_1'\oplus H_2'\oplus\cdots$, a contradiction.

LEMMA 4. *Let $G=\sum_J A_j$, J an arbitrary index set, A_j torsion complete. Let H be a torsion complete summand of G. Then $A_j=A_j^1\oplus A_j^2$, $H\cong\sum_J A_j^1$, $G/H\cong\sum_J A_j^2$.*

PROOF. We shall say that an arbitrary summand H of G can be *pulled out* of $\sum_J A_j$ if the A_j's decompose as above. The proof divides into three parts.

(1) If $H \subseteq \sum_I A_j$, I a finite subset of J, then H can be pulled out. It is clearly enough to show that H can be pulled out of $\sum_I A_j$. Since everything in sight is torsion complete, this follows from Corollary 34.5 of [2].

(2) If H is bounded, H can be pulled out. This is immediate upon examining the appropriate maximal p^n-bounded summands and their complements (which are necessarily isomorphic, see [1]).

(3) Let I be as in Lemma 3, T a pure subgroup of $\sum_I A_j$ such that $T[p] = H[p] \cap \sum_I A_j$ (Lemma 1), and $K = \prod_H T$. We note, as in the proof of Theorem 1, that T is torsion complete and isomorphic to K which is pure in H. Now $H = K \oplus H_0$ where H_0 is bounded. Therefore $G/H \oplus K \oplus H_0 \cong G$ and we may assume (replacing K by T using Theorem 17 of [3]) that $K \subseteq \sum_I A_j$. Thus we can pull K and then H_0 out and hence can pull out H.

We are now in a position to prove the bulk of the refinement theorem.

THEOREM 2. *Let G be a direct sum of countably many torsion complete groups. Any two decompositions of G into countably many groups have isomorphic refinements.*

PROOF. Suppose $G \cong \sum_1^\infty A_i \cong \sum_1^\infty B_j$. By Theorem 1 we can assume that A_i and B_j are torsion complete. We shall construct groups A_{ij} and B_{ij} such that $A_i \cong \sum_j (A_{ij} \oplus B_{ij})$ and $B_j \cong \sum_i (A_{ij} \oplus B_{ij})$. In the interest of clarity we shall not formalize the construction but give the first few steps and indicate the general procedure. By Lemma 4 we can pull A_1 out of $\sum_1^\infty B_j$; $A_1 = \sum_1^\infty A_{1j}$, $B_j \cong A_{1j} \oplus B_j'$, leaving $\sum_2^\infty A_i \cong \sum_1^\infty B_j'$. We now pull B_1' out of $\sum_2^\infty A_i$; $B_1' = \sum_2^\infty B_{i1}$, $A_i \cong B_{i1} \oplus A_i'$, leaving $\sum_2^\infty A_i' \cong \sum_2^\infty B_j'$. Now pull A_2' out of $\sum_2^\infty B_j'$ etc., crisscrossing back and forth and exhausting both decompositions.

We note the following corollary concerning semicomplete groups which follows immediately.

COROLLARY. *If G is a semicomplete group so is any summand of G.*

We now have the refinement theorem for countable decompositions. The next lemma, which parallels well known facts about decomposing torsion complete groups, says that all decompositions are essentially countable.

LEMMA 5. *Let G be a direct sum of countably many torsion complete groups. If* $G=\sum B_j$ *then all but a countable number of the* B_j*'s are bounded.*

PROOF. By Theorem 1 we may assume the B_j's are torsion complete, that there are uncountably many of them and that none are bounded. Let $G=\sum_1^\infty A_i$, A_i torsion complete. Since B_j is unbounded, there exists a positive integer n such that $B_j[p]\cap\sum_1^n A_i$ has unbounded height. Since there are uncountably many B_j's, there is some n for which this holds for infinitely many B_j's. This contradicts Lemma 3.

THEOREM 3. *Let G be a direct sum of countably many torsion complete groups. Any two decompositions of G have isomorphic refinements.*

PROOF. Let $G=\sum_I A_s=\sum_J B_t$. Theorem 1 and Lemma 5 allow us to assume that the A_s's and B_t's are torsion complete and all but a countable number of them are bounded. Relabeling, $\sum_I A_s = \sum_1^\infty A_i \oplus \sum_K C_s = \sum_J B_t = \sum_1^\infty B_j \oplus \sum_L D_t$, where each A_i and C_s is an A_s, the C_s's are bounded, and similarly for the B_j's and D_t's. We may further assume that each C_s and D_t is cyclic. Let $C=\sum C_s$, $D=\sum D_t$. By Theorem 2, the decompositions $\sum_1^\infty A_i\oplus C=\sum_1^\infty B_j\oplus D$ have isomorphic refinements $\sum E_u=\sum F_u$. We may assume that if E_u is a summand of C then $E_u=C_s$ for some s, and similarly for F_u. Thus each E_u is a summand of an A_s, $s\in I$, and each F_u is a summand of a B_t, $t\in J$, completing the proof.

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