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**$q$-ANALOGUES OF CAUCHY’S FORMULAS**

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1. Let $q$ be a given number and let $\alpha$ be real or complex. The $\alpha$th “basic number” is defined by means of $\lfloor \alpha \rfloor = (1-q^\alpha)/(1-q)$. This has served as a basis for an extensive amount of literature in mathematics under such titles as Heine, basic, or $q$-series and functions. The basic numbers also occur naturally in many theta identities. The works of Jackson (for bibliography see [2]) and Hahn [3] have stimulated much interest in this field.

One important operation that is intimately connected with basic series as well as with difference and other functional equations is the $q$-derivative of a function $f$. This is defined by

$$Df(x) = \frac{f(qx) - f(x)}{x(q - 1)}.$$  

Jackson defined the operations, which he called $q$-integration,

$$\int_0^x f(t) d(q, t) = x(1-q) \sum_{k=0}^{\infty} q^k f(xq^k)$$  

and

$$\int_x^{\infty} f(t) d(q, t) = x(1-q) \sum_{k=0}^{\infty} q^{-k} f(xq^{-k})$$

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provided the series on the right hand side are convergent. Both (1.2) and (1.3) are inverse operations to (1.1) and are analogues of the definite integrals \( \int_0^x f(t) \, dt \) and \( \int_x^a f(t) \, dt \) respectively. In fact each approach the respective integral as \( q \to 1 \) when \( f \) is Riemann integrable in the intervals of integration.

The definite \( q \)-integral \( \int_a^x f(t) \, dq(t) \) is defined by means of

\[
(1.4) \quad \int_a^x f(t) \, dq(t) = \int_0^x f(t) \, dq(t) - \int_0^a f(t) \, dq(t).
\]

The purpose of this note is to obtain \( q \)-analogues of Cauchy’s formulas for multiple integrals

\[
(1.5) \quad \int_a^x \int_a^{x_n-1} \cdots \int_a^{x_1} f(t) \, dt \, dx_1 \cdots dx_{n-1} = \frac{1}{(n-1)!} \int_a^x (x - t)^{n-1} f(t) \, dt
\]

and

\[
(1.6) \quad \int_x^\infty \int_x^{x_n-1} \cdots \int_x^{x_1} f(t) \, dt \, dx_1 \cdots dx_{n-1} = \frac{1}{(n-1)!} \int_x^\infty (t - x)^{n-1} f(t) \, dt.
\]

The \( q \)-analogues of (1.4) and (1.5) are given in Theorems 1 and 2 below. We remark that (3.1) and (3.2) can be regarded as transformations of \( n \)-fold infinite series to a single infinite series. These are basically different from a finite analogue of (1.5) that has been recently given by Traub [4]. We further note that (3.1) and (3.2) can be used to define fractional \( q \)-integral and \( q \)-derivative in the same way that (1.4) and (1.5) have been used to define fractional integrals and derivative. This we shall give elsewhere.

2. Let for a nonnegative integer \( n \), \([0]! = 1, [n]! = [n][n-1] \cdots [1]\) if \( n > 0 \), and let further, \( 0 < q < 1 \) and

\[
(a + b)_0 = 1, \quad (a + b)_n = (a + b)(a + bq) \cdots (a + bq^{n-1}) \quad \text{if} \quad n > 0.
\]

We shall also use the notation

\[
(a)_0 = 1, \quad (a)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) \quad \text{if} \quad n > 0,
\]

and
We first give the following lemmas:

**Lemma 1.** For integers \( n \geq 1 \) and \( m \geq 0 \) we have

\[
A_m = \sum_{j=0}^{m} q^j(q^j - q^{m+1})_{n-1} = \frac{(q^{m+1})_n}{1 - q^n}.
\]

**Proof.** We have

\[
\sum_{m=0}^{\infty} A_m u^m = \sum_{m=0}^{\infty} u^m \sum_{j=0}^{\infty} q^j(q^j - q^{m+1})_{n-1}
\]

\[
= \sum_{j=0}^{\infty} u^j q^n \sum_{m=0}^{\infty} (q^{m+1})_{n-1} u^m
\]

\[
= \frac{1}{1 - uq^n} \sum_{m=0}^{\infty} \frac{(q)_{m+1} (q)_m}{(q)_m} u^m,
\]

so we have

\[
\sum_{m=0}^{\infty} A_m u^m = \frac{(q)_{n-1}}{1 - uq^n} \sum_{m=0}^{\infty} \frac{(q^n)_m}{(q)_m} u^m.
\]

Let us recall the formula [1, p. 66]

\[
\sum_{k=0}^{\infty} \frac{(a)_k}{(q)_k} u^k = \prod_{r=0}^{\infty} \frac{1 - auq^r}{1 - uq^r},
\]

so that (2.2) can now be written as

\[
\sum_{m=0}^{\infty} A_m u^m = \frac{(q)_{n-1}}{1 - uq^n} \prod_{r=0}^{\infty} \frac{1 - uq^{r+n}}{1 - uq^r}
\]

\[
= (q)_{n-1} \prod_{r=0}^{\infty} \frac{1 - uq^{n+r+1}}{1 - uq^r} = (q)_{n-1} \sum_{k=0}^{\infty} \frac{(q^{n+1})_k}{(q)_k} u^k.
\]

Equating coefficients of \( u^m \) we get (2.1).

**Lemma 2.** For integral \( m \geq 1 \) we have

\[
\sum_{j=1}^{m} q^{-j}(q^{-m-1} - q^{-j})_{n-1} = \frac{q^{-n(m+1)+1}}{1 - q^n} (q^m)_n.
\]

The proof of this lemma is similar to that of Lemma 1.
Lemma 3. For nonnegative integer \( n \),

\[
a^{n+1}(1 - q^n) \sum_{j=0}^{\infty} q^j(q^j - q^{k+1})_{n-1}
\]

(2.4)

\[
- ax(1 - q^n) \sum_{j=1}^{\infty} q^j(xq^j - aq^{k+1})_{n-1} = a^{n+1}(q^{k+1})_n
\]

\[
= - a(x - aq^{k+1})_n.
\]

Proof. To prove this lemma we recall the identity due to Euler

(2.5)

\[
(a + b)_n = \sum_{r=0}^{n} (-1)^{r} \binom{n}{r} q^{r(r-1)/2} a^{n-r} b^r,
\]

so that

\[
\sum_{j=0}^{\infty} q^j(q^j - q^{k+1})_{n-1} = \sum_{r=0}^{n-1} (-1)^{r} \binom{n-1}{r} q^{r(r-1)/2} \frac{q^{r(k+1)}}{1 - q^{n-r}}.
\]

Hence we get

\[
a^{n+1}(1 - q^n) \sum_{j=0}^{\infty} q^j(q^j - q^{k+1})_{n-1}
\]

(2.6)

\[
= a^{n+1} \sum_{r=0}^{n-1} (-1)^{r} \binom{n-1}{r} q^{r(r-1)/2} q^{r(k+1)}
\]

\[
= a^{n+1}(1 - q^{k+1})_n - (-1)^n q^n(n-1)/2 + n(k+1).
\]

Similarly

\[
a x(1 - q^n) \sum_{j=0}^{\infty} q^j(xq^j - aq^{k+1})_{n-1}
\]

(2.7)

\[
= a[(x - aq^{k+1})_n - (-1)^n q^{(1/2)n(n-1)+n(k+1)} a^n].
\]

Substituting (2.6) and (2.7) in the left hand side of (2.4) we get the right hand side of (2.4).

3. We now prove our main results.

Theorem 1. If \( n \geq 1 \) is a given integer

\[
I^n f(x) = \int_{a}^{x} \int_{a}^{x_{n-1}} \cdots \int_{a}^{x_1} f(t) d(q, t)d(q, x_1) \cdots d(q, x_{n-1})
\]

(3.1)

\[
= \frac{1}{[n - 1]!} \int_{a}^{x} (x - qt)_{n-1} f(t) d(q, t),
\]
and

**Theorem 2.** If \( n \geq 1 \) is an integer

\[
K^nf(x) = \int_a^\infty \cdots \int_{x_{n-1}}^\infty f(t)d(q, t)d(q, x_1) \cdots d(q, x_{n-1})
\]
(3.2)

\[
= \frac{q^{-(1/2)n(n-1)}}{[n-1]!} \int_a^\infty (t-x)^{n-1}f(tq^{1-n})d(q, t).
\]

Both of these theorems can be proved by induction. We give the proof of the first as the proof of the second is similar. To prove Theorem 1 assume (3.1) is true for \( n = N \). We have

\[
I^{N+1}f(x) = \frac{1-q}{[N-1]!} \int_a^\infty \sum_{k=0}^\infty q^k \{ t(t - tq^{k+1})_{N-1}f(tq^k) - a(t - aq^{k+1})_{N-1}f(aq^k) \} d(q, t)
\]

\[
= \frac{(1-q)^2}{[N-1]!} \sum_{k=0}^\infty \sum_{j=0}^\infty q^{k+j}(1 - q^{k+1})_{N-1}\left\{ x^{N+1}q^{N+1}f(xq^{k+j}) - a^{N+1}q^{N+1}f(aq^{k+j}) \right\}
\]

\[
- \frac{(1-q)^2}{[N-1]} \sum_{k=0}^\infty \sum_{j=0}^\infty q^{k+j}f(aq^k) \{ ax(xq^j - aq^{k+1})_{N-1} - a^{N+1}(q^j - q^{k+1})_{N-1} \}.
\]

This reduces after some simplification to

\[
[N-1]!I^{N+1}f(x) = (1-q)^2 \sum_{s=0}^\infty q^sf(xq^s)(x^{N+1} - a^{N+1}) \sum_{j=0}^\infty q^j(q^j - q^{s+1})_{N-1}
\]

\[
- (1-q)^2 \sum_{s=0}^\infty q^sf(aq^s)ax \sum_{j=0}^\infty q^j(xq^j - aq^{s+1})_{N-1}
\]

\[
+ (1-q^2) \sum_{s=0}^\infty q^sf(aq^s)a^{N+1} \sum_{j=0}^\infty q^j(q^j - q^{s+1})_{N-1}.
\]

Evaluating the inside sums in the right hand side of the above equation by means of Lemmas 1 and 3 we get

\[
I^{N+1}f(x) = \frac{(1-q)^2}{[N]!} \sum_{s=0}^\infty q^s \{ x(x - xq^{s+1})_{N}f(xq^s) - a(x - aq^{s+1})_{N}f(aq^s) \}
\]

\[
= \frac{1}{[N]!} \int_a^\infty (x - qt)_{N}f(t) d(q, t).
\]
Since (3.1) is true for \( n = 1 \) our proof is complete. We remark that Lemma 2 is required in the proof of Theorem 2.

**References**


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