SOME RELATIONS ASSOCIATED WITH AN EXTENSION OF KOSHLIAKOV'S FORMULA

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It is known that the following three relations are equivalent:

(1) Functional equation for $\zeta^2(s)$

$$\pi^{-s}\left[\Gamma\left(\frac{s}{2}\right)\right]^2\zeta^2(s) = \pi^{s-1}\left[\Gamma\left(\frac{1-s}{2}\right)\right]^2[\zeta(1-s)]^2;$$

(2) Koshliakov's formula

$$\gamma - \log\left(\frac{4\pi}{\tau}\right) + 4\sum_{n=1}^{\infty} d(n)K_0(2\pi\tau n)$$
$$= \tau^{-1}[\gamma - \log(4\pi\tau)] + 4\tau^{-1}\sum_{n=1}^{\infty} d(n)K_0\left(\frac{2\pi}{\tau}n\right);$$

(3) Voronoï's sum formula

$$-\frac{1}{4}f(0) + \sum_{n=1}^{\infty} d(n)f(n) = \int_0^{\infty}(2\gamma + \log x)f(x)\,dx$$
$$+ 4\sum_{n=1}^{\infty} d(n)\int_0^{\infty} f(x)\left[K_0(4\pi(nx)^{1/2}) - \frac{\pi}{2}Y_0(4\pi(nx)^{1/2})\right]dx.$$

In fact, these were obtained in [**5**] by specializing parameters in relations which were proved equivalent. In the present note, we give a proof of the statement [**5**, p. 63] that two other special cases of (3), namely

$$\frac{\rho}{2\pi^2}\sum_{n=1}^{\infty} d(n)\frac{\log\left(\frac{n}{\rho}\right)}{n^2-\rho^2} = \frac{\gamma}{4} + \frac{1}{8}\log\rho + \frac{1}{8\pi^2\rho}\log(4\pi^2\rho) + \sum_{n=1}^{\infty} d(n)K_0(4\pi(n\rho)^{1/2}) \tag{4}$$

and

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$$\rho\sum_{n=1}^{\infty}\frac{d(n)}{\rho^2+n^2}=\frac{1}{4\rho}+\gamma\pi+\frac{1}{2}\pi\log\rho \tag{5}$$
$$+2\pi\sum_{n=1}^{\infty}d(n)\{K_0(4\pi(-in\rho)^{1/2})+K_0(4\pi(in\rho)^{1/2})\}$$

can also be considered equivalent to (3). Thus any one of the relations (1) through (5) implies the others. However, as in [**5**], we shall treat the problem in a somewhat generalized form.

In the course of our investigation, we shall need the following results:

$$\int_0^\infty x^{\rho-1}K_0(xy)\,dx=\frac{1}{4}\left(\frac{2}{y}\right)^{\rho}\left[\Gamma\left(\frac{\rho}{2}\right)\right]^2, \tag{6}$$ [**1**, p. 127],

$$\int_0^\infty K_0\left(\frac{a}{x}\right)K_0(xy)\,dx=\frac{\pi}{y}K_0(2(ay)^{1/2}), \tag{7}$$ [**1**, p. 146],

$$\int_0^\infty xK_0(ax)K_0(xy)\,dx=\frac{\log\left(\frac{a}{y}\right)}{a^2-y^2}, \tag{8}$$

$$\int_0^\infty \log x\, x^{\rho-1}K_0(xy)\,dx \tag{9}$$
$$=\frac{1}{4}\left(\frac{2}{y}\right)^{\rho}\left[\Gamma\left(\frac{\rho}{2}\right)\right]^2\left\{\log\left(\frac{2}{y}\right)+\psi\left(\frac{\rho}{2}\right)\right\}.$$

(8) can easily be obtained from [**1**, p. 145] and (9) from (6) above.

$$\int_0^\infty K_0(ax)\left[\frac{2}{\pi}K_0((xy)^{1/2})-Y_0((xy)^{1/2})\right]dx=\frac{1}{a}K_0\left(\frac{y}{4a}\right) \tag{9a}$$

[**5**, p. 51].

1. We assume that

$$\begin{aligned}&a_1, a_2\cdots; \qquad 0<\lambda_1<\lambda_2\cdots\to\infty,\\ &b_1, b_2\cdots; \qquad 0<l_1<l_2\cdots\to\infty\end{aligned} \tag{a}$$

are four sequences of numbers such that the series

$$\sum_{n=1}^{\infty}\frac{|a_n|}{\lambda_n^2} \quad\text{and}\quad \sum_{n=1}^{\infty}\frac{|b_n|}{l_n^2}$$

are convergent.

(b) α, α', β, β' are four numbers such that the following relation holds

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$$\text{(I)}\quad \begin{aligned}&\alpha[\gamma + \log(\pi\tau)] - \alpha' + \sum_{n=1}^{\infty} a_n K_0(2\pi\lambda_n\tau)\\ &\qquad = \frac{1}{\tau}\left\{\beta\left[\gamma + \log\left(\frac{\pi}{\tau}\right)\right] - \beta' + \sum_{n=1}^{\infty} b_n K_0\left(\frac{2\pi l_n}{\tau}\right)\right\}.\end{aligned}$$

If we multiply both sides of (I) by $\tau K_0(2\pi\tau y)$, $(y>0)$, and integrate with regard to τ from zero to infinity, we get

$$\text{(10)}\quad \begin{aligned}&[\alpha\gamma + \alpha\log\pi - \alpha']\frac{1}{4\pi^2y^2} + \alpha\frac{1}{4\pi^2y^2}[-\log\pi y - \gamma]\\ &\qquad + \frac{1}{4\pi^2}\sum_{n=1}^{\infty} a_n \frac{\log\left(\frac{\lambda_n}{y}\right)}{\lambda_n^2 - y^2}\\ &\qquad = [\beta\gamma + \beta\log\pi - \beta']\frac{1}{4y} + \frac{\beta}{4y}[\gamma + \log(4\pi y)]\\ &\qquad + \frac{1}{2y}\sum_{n=1}^{\infty} b_n K_0(4\pi(l_n y)^{1/2}).\end{aligned}$$

The interchange of the order of summation and integration is justified because of absolute convergence.

We rewrite (10) as

$$\text{(II)}\quad \begin{aligned}&\frac{1}{2\pi^2y}[-\alpha\log y - \alpha'] + \frac{y}{2\pi^2}\sum_{n=1}^{\infty} a_n \frac{\log\frac{\lambda_n}{y}}{\lambda_n^2 - y^2}\\ &\qquad = \left[\beta\gamma + \frac{1}{2}\beta\log(4\pi^2y) - \frac{1}{2}\beta'\right] + \sum_{n=1}^{\infty} b_n K_0(4\pi(l_n y)^{1/2}).\end{aligned}$$

By analytic continuation (II) holds for all y such that $-\pi<\arg y<\pi$. Likewise, if we multiply (I) by $(1/\tau^2)K_0(2\pi y/\tau)$ and integrate, we get

$$\text{(II}'\text{)}\quad \begin{aligned}&\left(\alpha\gamma + \frac{1}{2}\alpha\log(4\pi^2y) - \frac{1}{2}\alpha'\right) + \sum_{n=1}^{\infty} a_n K_0(4\pi(\lambda_n y)^{1/2})\\ &\qquad = \frac{1}{2\pi^2y}(-\beta\log y - \beta') + \frac{y}{2\pi^2}\sum_{n=1}^{\infty} b_n \frac{\log\left(\frac{l_n}{y}\right)}{l_n^2 - y^2}.\end{aligned}$$

If we replace y by $e^{i\pi}y$ in (II), we get the relation

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$$\frac{1}{2\pi^2 y}\left[\alpha \log (ye^{i\pi}) + \alpha'\right] + \frac{y}{2\pi^2}\sum_{n=1}^{\infty} a_n \frac{\log\left(\dfrac{\lambda_n}{ye^{i\pi}}\right)}{y^2-\lambda_n^2}$$

$$(11) \qquad = \left[\beta\gamma + \frac{1}{2}\beta \log (4\pi^2 y e^{i\pi}) - \frac{1}{2}\beta'\right] + \sum_{n=1}^{\infty} b_n K_0(4\pi(l_n y e^{i\pi})^{1/2}),$$

$$-2\pi < \arg y < 0.$$

Adding the corresponding sides of (II) and (11), we obtain

$$\frac{i\alpha}{2\pi y} - \frac{iy}{2\pi}\sum_{n=1}^{\infty}\frac{a_n}{y^2-\lambda_n^2}$$

$$(\mathrm{III}) \quad = 2\beta\gamma + \beta\log(4\pi^2 y) - \beta' + \tfrac{1}{2}i\pi\beta$$

$$+ \sum_{n=1}^{\infty} b_n\left\{K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{i\pi})^{1/2})\right\}, \quad -\pi < \arg y < 0.$$

The corresponding relation derived from (II′) is

$$2\alpha\gamma + \alpha\log(4\pi^2 y) - \alpha' + \tfrac{1}{2}i\pi\alpha$$

$$(\mathrm{III}') \qquad + \sum_{n=1}^{\infty} a_n\left\{K_0(4\pi(\lambda_n y)^{1/2}) + K_0(4\pi(\lambda_n y e^{i\pi})^{1/2})\right\}$$

$$= \frac{i\beta}{2\pi y} - \frac{iy}{2\pi}\sum_{n=1}^{\infty}\frac{b_n}{y^2-l_n^2}, \qquad -\pi < \arg y < 0.$$

2. Let

$$\sigma(y) = \frac{1}{2\pi i}\left\{\sum_{n=1}^{\infty} a_n \frac{2y}{y^2-\lambda_n^2} - 2\frac{\alpha}{y}\right\}.$$

Then from (III)

$$\tfrac{1}{2}\sigma(y) = 2\beta\gamma + \beta\log(4\pi^2 y) - \beta' + \tfrac{1}{2}i\pi\beta$$

$$+ \sum_{n=1}^{\infty} b_n\left\{K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{i\pi})^{1/2})\right\}, \quad -\pi < \arg y < 0,$$

$$= -\left[2\beta\gamma + \beta\log(4\pi^2 y) - \beta'\right] + \tfrac{1}{2}i\pi\beta$$

$$- \sum_{n=1}^{\infty} b_n\left\{K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{-i\pi})^{1/2})\right\}, \quad 0 < \arg y < \pi.$$

Let $\phi(y)$ be a function of the complex variable $y = u + iv$, regular in a strip $u \geqq 0$, $|v| \leqq \delta$, for some $\delta > 0$ and satisfying the following conditions:

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(i) $\int_0^\infty |\phi(u+iv)|\,du$ and $\int_0^\infty \log(u+iv)\,\phi(u+iv)du$ converge in

(12)

$$-\delta<v<\delta.$$

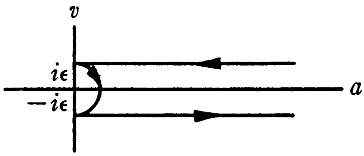
(ii) There exists a sequence of the numbers u_n such that

(13) $\lim_{n\to\infty} \sigma(u_n+iv)\phi(u_n+iv) = 0$ uniformly in $-\delta<v<\delta$.

Since $\sigma(y)$ is analytic except for simple poles at zero and $\pm\lambda_n$, $n=1, 2, 3, \cdots$, we obtain by Cauchy's theorem and (13)

$$\sum_{n=1}^\infty a_n\phi(\lambda_n) = \int_C \sigma(y)\phi(y)dy$$

where C is the contour shown. Now we let $\epsilon\to 0$.



$$0<\epsilon<\min(\delta,\lambda_1).$$

Thus

$$\begin{aligned}\sum_{n=1}^\infty a_n\phi(\lambda_n) &= \alpha\phi(0) + 4\int_0^\infty [2\beta\gamma - \beta' + \beta\log(4\pi^2 u)]\phi(u)\,du \\ &+ \lim_{\epsilon\to 0}\left\{2\int_{-i\epsilon}^{\infty-i\epsilon}\phi(y)\sum_{n=1}^\infty b_n[K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{i\pi})^{1/2})]\,dy\right. \\ &\left.+2\int_{i\epsilon}^{\infty+i\epsilon}\phi(y)\sum_{n=1}^\infty b_n[K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{-i\pi})^{1/2})]\,dy\right\}.\end{aligned}$$

Let $\phi(y)$ be such that we can interchange the order of summation and integration. Then proceeding to the limit as $\epsilon\to 0$; and using

$$K_0(ye^{i\pi/2}) + K_0(ye^{-i\pi/2}) = -\pi Y_0(y),$$

we obtain

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$$\text{(IV)}\quad \begin{aligned} -\alpha\phi(0) + \sum_{n=1}^{\infty} a_n\phi(\lambda_n) &= 4\int_0^{\infty}[2\beta\gamma - \beta' + \beta\log(4\pi^2 u)]\phi(u)\,du \\ &+ 4\sum_{n=1}^{\infty} b_n \int_0^{\infty}\left[K_0(4\pi(l_n u)^{1/2}) - \frac{\pi}{2}Y_0(4\pi(l_n u)^{1/2})\right]\phi(u)\,du. \end{aligned}$$

Likewise from (III′), we get the Sum-formula

$$\text{(IV}'\text{)}\quad \begin{aligned} -\beta\phi(0) + \sum_{n=1}^{\infty} b_n\phi(l_n) &= 4\int_0^{\infty}[2\alpha\gamma - \alpha' + \alpha\log(4\pi^2 u)]\phi(u)\,du \\ &+ 4\sum_{n=1}^{\infty} a_n \int_0^{\infty}\left[K_0(4\pi(\lambda_n u)^{1/2}) - \frac{\pi}{2}Y_0(4\pi(\lambda_n u)^{1/2})\right]\phi(u)\,du. \end{aligned}$$

If in (IV) or (IV′), we let

$$\phi(u) = K_0(2\pi u\tau) - K_0(2\pi u), \qquad 0 < u < \infty,$$
$$\phi(0) = \log\frac{1}{\tau},$$

and further assume that

$$\text{(c)}\quad \begin{aligned} &\alpha[\gamma + \log\pi] - \alpha' + \sum_{n=1}^{\infty} a_n K_0(2\pi\lambda_n) \\ &\qquad = \beta[\gamma + \log\pi] - \beta' + \sum_{n=1}^{\infty} b_n K_0(2\pi l_n). \end{aligned}$$

Then using (6), (9), (9a), we get the relation (I).

Thus we have proved the following:

1. Under condition (a), each one of the six relations (II), (II′), (III), (III′), (IV), (IV′) is a consequence of (I).

2. Under conditions (a) and (c), the relation (I) through (IV′) are equivalent to each other.

If we set

$$a_n = b_n = d(n); \qquad \lambda_n = l_n = n,$$
$$\alpha = \beta = \tfrac{1}{4}; \qquad \alpha' = \beta' = \tfrac{1}{2}\log(2\pi),$$

in relations (I)–(IV′) and note that conditions (a) and (c) are satisfied, we get the corresponding special cases mentioned in the introduction.

3. Now we study the functions $f(s)$ and $g(s)$ defined by:

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$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \qquad \operatorname{Re} s \geqq 2,$$

$$g(s) = \sum_{n=1}^{\infty} \frac{b_n}{l_n^2}, \qquad \operatorname{Re} s \geqq 2,$$

when conditions (a) and (b) hold.

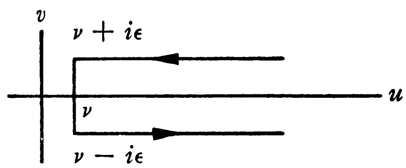
Let

$$\mu(y) = \sum_{n=1}^{\infty} b_n \{ K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n y e^{i\pi})^{1/2}) \}, \qquad -\pi < \arg y < 0,$$

then

$$\mu(e^{-i\pi}y) = \sum_{n=1}^{\infty} b_n \{ K_0(4\pi(l_n y e^{-i\pi})^{1/2}) + K_0(4\pi(l_n y)^{1/2}) \}, \quad 0 < \arg y < \pi.$$

We denote the contour opposite as C_ν.



For $0 < \nu < \lambda_1$ and $\phi(y) = y^{-s}$, Re $s \geqq 2$, we obtain

$$\begin{aligned} f(s) &= \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} = \int_{C_\nu} \sigma(y) y^{-s}\, dy \\ &= 4\int_{\nu}^{\infty} [2\beta\gamma - \beta' + \beta \log (4\pi^2 u)] u^{-s}\, du + 2\int_{\nu}^{\nu - i\epsilon} \frac{\mu(y)}{y^s}\, dy \\ &\quad + 2\int_{\nu}^{\nu+i\epsilon} \frac{\mu(e^{-i\pi}y)}{y^s}\, dy + 2\int_{\nu-i\epsilon}^{\infty - i\epsilon} \frac{\mu(y)}{y^s}\, dy + 2\int_{\nu+i\epsilon}^{\infty+i\epsilon} \frac{\mu(e^{-i\pi}y)}{y^s}\, dy. \end{aligned}$$

Making use of the fact that

$$\sum_{n=1}^{\infty} b_n K_0(4\pi(l_n z)^{1/2}) = O(\exp[-4\pi(l_1\rho)^{1/2} \cos \theta/2)$$

where $z = \rho e^{i\theta}$, we replace the path of integration in the last two integrals by ones at right-angles to them. Thus

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$$f(s) = 4[2\beta\gamma - \beta' + \beta \log (4\pi^2)] \frac{\nu^{1-s}}{s-1} + 4\beta \log \nu \frac{\nu^{1-s}}{s-1} \tag{14}$$
$$+ 4\beta \frac{\nu^{1-s}}{(s-1)^2} + 2 \int_\nu^{\nu - i\infty} \frac{\mu(y)}{y^s} dy + 2 \int_\nu^{\nu + i\infty} \frac{\mu(e^{-i\pi}y)}{y} dy.$$

But the above integrals define a function of s which is regular for all finite values of s. Therefore $f(s)$ is analytic and single valued in the whole s-plane except perhaps at $s=1$, where it may have a pole of the second order, with the principal part

$$\frac{8\beta\gamma - 4\beta' + 4\beta \log (4\pi^2)}{s-1} + \frac{4\beta}{(s-1)^2}.$$

Likewise, using (III′), which is a consequence of (b), we can show that $g(s)$ has the same properties as $f(s)$, and that the principal part of $g(s)$ at $s=1$ is

$$\frac{4\alpha}{(s-1)^2} + \frac{8\alpha\gamma - 4\alpha' + 4\alpha \log (4\pi^2)}{s-1}.$$

Now if we proceed to the limit as $\nu \to 0$ in $\operatorname{Re} s < 1$, we obtain from (14)

$$\begin{aligned} f(s) &= 2 \int_0^{-i\infty} \frac{\mu(y)}{y^s} dy + 2 \int_0^{i\infty} \frac{\mu(e^{-i\pi}y)}{y^s} dy \\ &= 4 \sin\left(\frac{\pi}{2} s\right) \int_0^\infty \frac{\mu(e^{-i\pi/2}t)}{t^s} dt. \end{aligned} \tag{15}$$

This gives the integral representation of $f(s)$ in the half s-plane.

If we interchange the order of integration and summation in (15) and use

$$\int_0^\infty K_0(at^{1/2}) t^{-s} dt = a^{2s-2} 2^{1-2s} [\Gamma(1-s)]^2, \quad \operatorname{Re} s < 1; \ |\arg a| < \frac{\pi}{2},$$

we obtain

$$f(s) = \left[\sin\left(\frac{\pi}{2} s\right)\right]^2 [\Gamma(1-s)]^2 2^{2s-2} g(1-s) \pi^{2s-2}, \qquad \operatorname{Re} s < -1,$$

which can be written as

$$\pi^{-s} \left[\Gamma\left(\frac{s}{2}\right)\right]^2 f(s) = \pi^{-1+s} \left[\Gamma\left(\frac{1-s}{2}\right)\right]^2 g(1-s) \tag{16}$$

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and by analytic continuation equality (16) holds for all s.

Finally, expanding both sides of (16) around $s=0$ and $s=1$ respectively, we find that

$$\alpha = f(0), \quad \alpha' = f'(0), \quad \beta = g(0), \quad \beta' = g'(0). \tag{17}$$

Thus we have proved the following: Under conditions (a) and (b), $(s-1)^2 f(s)$ and $(s-1)^2 g(s)$ are entire functions; $f(s)$ and $g(s)$ satisfy the functional equation (16) and further α, α', β, β' are related to these functions as given in (17).

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REFERENCES

1. A. Erdélyi, *Tables of integral transforms*, Vol. 2, McGraw-Hill, New York, 1954.

2. H. Hamburger, *Ueber einige Beziehungen die mit der Funktional gleichung der Zeta Funktion aequivalent Sind*, Math. Ann. **85** (1922), 129–140.

3. N. S. Koshliakov, *Application of the theory of sum formulae to the investigation of a class of one valued analytical functions in the theory of numbers*, Messenger of Math. **58** (1928), 1–23.

4. ———, *On Voronoĭ's sum formula*, Messenger of Math. **58** (1928), 30–32.

5. K. Soni, *On some relations which are equivalent to the functional equations of the type of Riemann's zeta function and tables of divisor transforms*, Ph.D. Thesis, Oregon State University, Corvallis, Oregon, 1964.

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