SOME RELATIONS ASSOCIATED WITH AN EXTENSION OF KOSHLIAKOV'S FORMULA

KUSUM SONI

It is known that the following three relations are equivalent:

1. Functional equation for $\xi^2(s)$

$$\pi^{-s} \left[ \Gamma \left( \frac{s}{2} \right) \right]^2 \xi^2(s) = \pi^{s-1} \left[ \Gamma \left( \frac{1-s}{2} \right) \right]^2 \xi(1-s)^2;$$

2. Koshliakov's formula

$$\gamma - \log \left( \frac{4\pi}{\tau} \right) + 4 \sum_{n=1}^{\infty} d(n) K_0(2\pi\tau n) \quad = \quad \tau^{-1} \left[ \gamma - \log(4\pi\tau) \right] + 4\tau^{-1} \sum_{n=1}^{\infty} d(n) K_0 \left( \frac{2\pi}{\tau} n \right);$$

3. Voronoi's sum formula

$$- \frac{1}{4} f(0) + \sum_{n=1}^{\infty} d(n)f(n) = \int_{0}^{\infty} (2\gamma + \log x)f(x) \, dx$$

$$+ 4 \sum_{n=1}^{\infty} d(n) \int_{0}^{\infty} f(x) \left[ K_0(4\pi(nx)^{1/2}) - \frac{\pi}{2} V_0(4\pi(nx)^{1/2}) \right] \, dx.$$ 

In fact, these were obtained in [5] by specializing parameters in relations which were proved equivalent. In the present note, we give a proof of the statement [5, p. 63] that two other special cases of (3), namely

$$\frac{\rho}{2\pi^2} \sum_{n=1}^{\infty} d(n) \frac{\log \left( \frac{n}{\rho} \right)}{n^2 - \rho^2} = \frac{\gamma}{4} + \frac{1}{8} \log \rho + \frac{1}{8\pi^2\rho} \log (4\pi^2\rho)$$

$$+ \sum_{n=1}^{\infty} d(n) K_0(4\pi(n\rho)^{1/2})$$

and

Presented to the Society, September 3, 1965 under the title Some relations associated with Koshliakov's formula; received by the editors February 1, 1965.
\[ \rho \sum_{n=1}^{\infty} \frac{d(n)}{\rho^2 + n^2} = \frac{1}{4\rho} + \gamma \pi + \frac{1}{2} \pi \log \rho \]

(5)

\[ + 2\pi \sum_{n=1}^{\infty} d(n) \left\{ K_0(4\pi(-in\rho)^{1/2}) + K_0(4\pi(in\rho)^{1/2}) \right\}, \]

can also be considered equivalent to (3). Thus any one of the relations (1) through (5) implies the others. However, as in [5], we shall treat the problem in a somewhat generalized form.

In the course of our investigation, we shall need the following results:

(6) \[ \int_0^\infty x^{\sigma-1} K_0(xy) \, dx = \frac{1}{4} \left( \frac{2}{y} \right)^\rho \left[ \Gamma \left( \frac{\rho}{2} \right) \right]^2, \] [1, p. 127],

(7) \[ \int_0^\infty K_0 \left( \frac{a}{x} \right) K_0(xy) \, dx = \frac{\pi}{y} K_0(2(ay)^{1/2}), \] [1, p. 146],

(8) \[ \int_0^\infty xK_0(ax)K_0(xy) \, dx = \frac{\log \left( \frac{a}{y} \right)}{a^2 - y^2}, \]

(9) \[ \int_0^\infty \log x \, x^{\sigma-1} K_0(xy) \, dx = \frac{1}{4} \left( \frac{2}{y} \right)^\rho \left[ \Gamma \left( \frac{\rho}{2} \right) \right]^2 \left\{ \log \left( \frac{2}{y} \right) + \psi \left( \frac{\rho}{2} \right) \right\}. \]

(8) can easily be obtained from [1, p. 145] and (9) from (6) above.

(9a) \[ \int_0^\infty K_0(ax) \left[ \frac{2}{\pi} K_0((xy)^{1/2}) - Y_0((xy)^{1/2}) \right] \, dx = \frac{1}{a} K_0 \left( \frac{y}{4a} \right) \] [5, p. 51].

1. We assume that

(a) \[ a_1, a_2, \ldots ; \quad 0 < \lambda_1 < \lambda_2 \ldots \to \infty, \]

(b) \[ b_1, b_2, \ldots ; \quad 0 < l_1 < l_2 \ldots \to \infty \]

are four sequences of numbers such that the series

\[ \sum_{n=1}^{\infty} \frac{|a_n|}{\lambda_n^2} \] \[ \text{and} \] \[ \sum_{n=1}^{\infty} \frac{|b_n|}{l_n^2} \]

are convergent.

(b) \[ \alpha, \alpha', \beta, \beta' \] are four numbers such that the following relation holds
\[
\alpha[\gamma + \log(\pi\tau)] - \alpha' + \sum_{n=1}^{\infty} a_n K_0(2\pi\lambda_n\tau)
\]

(I)

\[
= \frac{1}{\tau} \left\{ \beta \left[ \gamma + \log\left(\frac{\pi}{\tau}\right) \right] - \beta' + \sum_{n=1}^{\infty} b_n K_0\left(\frac{2\pi l_n}{\tau}\right) \right\}.
\]

If we multiply both sides of (I) by \(\tau K_0(2\pi\gamma y), (y > 0)\), and integrate with regard to \(\tau\) from zero to infinity, we get

\[
[\alpha\gamma + \alpha \log \pi - \alpha'] \frac{1}{4\pi^2 y^2} + \alpha \frac{1}{4\pi^2 y^2} \left[ -\log \pi y - \gamma \right]
\]

\[
+ \frac{1}{4\pi^2} \sum_{n=1}^{\infty} a_n \frac{\log\left(\frac{\lambda_n}{y}\right)}{\lambda_n^2 - y^2}
\]

\[
= \left[ \beta\gamma + \beta \log \pi - \beta' \right] \frac{1}{4y} + \frac{\beta}{4y} \left[ \gamma + \log (4\pi y) \right]
\]

\[
+ \frac{1}{2y} \sum_{n=1}^{\infty} b_n K_0(4\pi(l_n y)^{1/2}).
\]

(10)

The interchange of the order of summation and integration is justified because of absolute convergence.

We rewrite (10) as

\[
\frac{1}{2\pi^2 y} \left[ -\alpha \log y - \alpha' \right] + \frac{y}{2\pi^2} \sum_{n=1}^{\infty} a_n \frac{\log\left(\frac{y}{\lambda_n^2 - y^2}\right)}{\lambda_n^2 - y^2}
\]

(II)

\[
= \left[ \beta\gamma + \frac{1}{2} \beta \log (4\pi^2 y) - \frac{1}{2} \beta' \right] + \sum_{n=1}^{\infty} b_n K_0(4\pi(l_n y)^{1/2}).
\]

By analytic continuation (II) holds for all \(y\) such that \(-\pi < \arg y < \pi\).

Likewise, if we multiply (I) by \((1/\tau^2)K_0(2\pi y/\tau)\) and integrate, we get

\[
\left( \alpha\gamma + \frac{1}{2} \alpha \log (4\pi^2 y) - \frac{1}{2} \alpha' \right) + \sum_{n=1}^{\infty} a_n K_0(4\pi(\lambda_n y)^{1/2})
\]

(II')

\[
= \frac{1}{2\pi^2 y} (-\beta \log y - \beta') + \frac{y}{2\pi^2} \sum_{n=1}^{\infty} b_n \frac{\log\left(\frac{l_n}{y}\right)}{l_n^2 - y^2}.
\]

If we replace \(y\) by \(e^{i\pi y}\) in (II), we get the relation
\[
\frac{1}{2\pi^2 y} \left[ \alpha \log ( ye^{i\pi}) + \alpha' \right] + \frac{y}{2\pi^2} \sum_{n=1}^{\infty} a_n \frac{\log \left( \frac{\lambda_n}{ye^{i\pi}} \right)}{y^2 - \lambda_n^2} \\
= \left[ \beta \gamma + \frac{1}{2} \beta \log (4\pi^2 ye^{i\pi}) - \frac{1}{2} \beta' \right] + \sum_{n=1}^{\infty} b_n K_0(4\pi (l_n ye^{i\pi})^{1/2}), \quad -2\pi < \arg y < 0.
\]

Adding the corresponding sides of (II) and (11), we obtain

\[
\frac{ia}{2\pi y} - \frac{iy}{2\pi} \sum_{n=1}^{\infty} \frac{a_n}{y^2 - \lambda_n^2} = 2\beta \gamma + \beta \log (4\pi^2 y) - \beta' + \frac{1}{2} i\pi \beta
\\
+ \sum_{n=1}^{\infty} b_n \left\{ K_0(4\pi (l_n y)^{1/2}) + K_0(4\pi (l_n ye^{i\pi})^{1/2}) \right\}, \quad -\pi < \arg y < 0.
\]

The corresponding relation derived from (II') is

\[
2\alpha \gamma + \alpha \log (4\pi^2 y) - \alpha' + \frac{1}{2} i\pi \alpha
\\
+ \sum_{n=1}^{\infty} a_n \left\{ K_0(4\pi (\lambda_n y)^{1/2}) + K_0(4\pi (\lambda_n ye^{i\pi})^{1/2}) \right\}
\\
= \frac{i\beta}{2\pi y} - \frac{iy}{2\pi} \sum_{n=1}^{\infty} \frac{b_n}{y^2 - l_n^2}, \quad -\pi < \arg y < 0.
\]

2. Let

\[
\frac{1}{2\pi i} \left\{ \sum_{n=1}^{\infty} a_n \frac{2y}{y^2 - \lambda_n^2} - 2 \frac{\alpha}{y} \right\}.
\]

Then from (III)

\[
\frac{1}{2} \sigma(y) = 2\beta \gamma + \beta \log (4\pi^2 y) - \beta' + \frac{1}{2} i\pi \beta
\\
+ \sum_{n=1}^{\infty} b_n \left\{ K_0(4\pi (l_n y)^{1/2}) + K_0(4\pi (l_n ye^{i\pi})^{1/2}) \right\}, \quad -\pi < \arg y < 0,
\\
= - \left[ 2\beta \gamma + \beta \log (4\pi^2 y) - \beta' \right] + \frac{1}{2} i\pi \beta
\\
- \sum_{n=1}^{\infty} b_n \left\{ K_0(4\pi (l_n y)^{1/2}) + K_0(4\pi (l_n ye^{-i\pi})^{1/2}) \right\}, \quad 0 < \arg y < \pi.
\]

Let \(\phi(y)\) be a function of the complex variable \(y = u + iv\), regular in a strip \(u \geq 0, |v| \leq \delta\), for some \(\delta > 0\) and satisfying the following conditions:
(i) $\int_0^\infty |\phi(u+iv)| \, du$ and $\int_0^\infty \log (u+iv) \phi (u+iv) \, du$ converge in $-\delta < v < \delta$. (12)

(ii) There exists a sequence of the numbers $u_n$ such that

$$\lim_{n \to \infty} \sigma(u_n + iv) \phi(u_n + iv) = 0$$ uniformly in $-\delta < v < \delta$. (13)

Since $\sigma(y)$ is analytic except for simple poles at zero and $\pm \lambda_n$, \( n = 1, 2, 3, \ldots \), we obtain by Cauchy’s theorem and (13)

$$\sum_{n=1}^\infty a_n \phi(\lambda_n) = \int_C \sigma(y) \phi(y) \, dy$$

where $C$ is the contour shown. Now we let $\epsilon \to 0$.

Thus

$$\sum_{n=1}^\infty a_n \phi(\lambda_n)$$

$$= \alpha \phi(0) + 4 \int_0^\infty [2\beta \gamma - \beta' + \beta \log (4\pi^2 u)] \phi(u) \, du$$

$$+ \lim_{\epsilon \to 0} \left\{ 2 \int_{-\infty-i\epsilon}^{\infty-i\epsilon} \phi(y) \sum_{n=1}^\infty b_n [K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n ye^{i\pi})^{1/2})] \, dy 

+ 2 \int_{i\epsilon}^{\infty+i\epsilon} \phi(y) \sum_{n=1}^\infty b_n [K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n ye^{-i\pi})^{1/2})] \, dy \right\}.$$

Let $\phi(y)$ be such that we can interchange the order of summation and integration. Then proceeding to the limit as $\epsilon \to 0$; and using

$$K_0(ye^{i\pi/2}) + K_0(ye^{-i\pi/2}) = -\pi Y_0(y),$$

we obtain
\[-\alpha \phi(0) + \sum_{n=1}^{\infty} a_n \phi(\lambda_n) = 4 \int_{0}^{\infty} \left[ 2\beta \gamma - \beta' + \beta \log (4\pi^2 u) \right] \phi(u) \, du \]

(IV)

\[+ 4 \sum_{n=1}^{\infty} b_n \int_{0}^{\infty} \left[ K_0(4\pi (l_n u)^{1/2}) - \frac{\pi}{2} Y_0(4\pi (l_n u)^{1/2}) \right] \phi(u) \, du.\]

Likewise from (III'), we get the Sum-formula

\[-\beta \phi(0) + \sum_{n=1}^{\infty} b_n \phi(l_n) = 4 \int_{0}^{\infty} \left[ 2\alpha \gamma - \alpha' + \alpha \log (4\pi^2 u) \right] \phi(u) \, du \]

(IV')

\[+ 4 \sum_{n=1}^{\infty} a_n \int_{0}^{\infty} \left[ K_0(4\pi (\lambda_n u)^{1/2}) - \frac{\pi}{2} Y_0(4\pi (\lambda_n u)^{1/2}) \right] \phi(u) \, du.\]

If in (IV) or (IV'), we let

\[\phi(u) = K_0(2\pi nu) - K_0(2\pi u), \quad 0 < u < \infty,\]

\[\phi(0) = \log \frac{1}{\tau},\]

and further assume that

\[\alpha [\gamma + \log \pi] - \alpha' + \sum_{n=1}^{\infty} a_n K_0(2\pi \lambda_n)\]

(c)

\[= \beta [\gamma + \log \pi] - \beta' + \sum_{n=1}^{\infty} b_n K_0(2\pi l_n).\]

Then using (6), (9), (9a), we get the relation (I).

Thus we have proved the following:

1. Under condition (a), each one of the six relations (II), (II'), (III), (III'), (IV), (IV') is a consequence of (I).

2. Under conditions (a) and (c), the relation (I) through (IV') are equivalent to each other.

If we set

\[a_n = b_n = d(n); \quad \lambda_n = l_n = n,\]

\[\alpha = \beta = \frac{1}{4}; \quad \alpha' = \beta' = \frac{1}{2} \log (2\pi),\]

in relations (I)–(IV') and note that conditions (a) and (c) are satisfied, we get the corresponding special cases mentioned in the introduction.

3. Now we study the functions \(f(s)\) and \(g(s)\) defined by:
\[ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s}, \quad \text{Re} \ s \geq 2, \]

\[ g(s) = \sum_{n=1}^{\infty} \frac{b_n}{\beta_n^2}, \quad \text{Re} \ s \geq 2, \]

when conditions (a) and (b) hold.

Let

\[ \mu(y) = \sum_{n=1}^{\infty} b_n \{ K_0(4\pi(l_n y)^{1/2}) + K_0(4\pi(l_n ye^{ix})^{1/2}) \}, \quad -\pi < \arg y < 0, \]

then

\[ \mu(e^{-ix}y) = \sum_{n=1}^{\infty} b_n \{ K_0(4\pi(l_n ye^{-ix})^{1/2}) + K_0(4\pi(l_n y)^{1/2}) \}, \quad 0 < \arg y < \pi. \]

We denote the contour opposite as \( C' \).

For \( 0 < v < \lambda_1 \) and \( \phi(y) = y^{-s}, \ \text{Re} \ s \geq 2, \) we obtain

\[ f(s) = \sum_{n=1}^{\infty} \frac{a_n}{\lambda_n^s} = \int_{C'} \sigma(y)y^{-s} \, dy \]

\[ = 4 \int_{v}^{\infty} [2\beta y - \beta' + \beta \log (4\pi^2 u)] u^{-s} \, du + 2 \int_{v}^{\infty+i} \frac{\mu(y)}{y^s} \, dy \]

\[ + 2 \int_{v}^{r+ie} \frac{\mu(e^{-ix}y)}{y^s} \, dy + 2 \int_{r-ie}^{\infty-i} \frac{\mu(y)}{y^s} \, dy + 2 \int_{r+ie}^{\infty+i} \frac{\mu(e^{-ix}y)}{y^s} \, dy. \]

Making use of the fact that

\[ \sum_{n=1}^{\infty} b_n K_0(4\pi(l_n z)^{1/2}) = O(\exp[-4\pi(l_1 \rho)^{1/2} \cos \theta/2]) \]

where \( z = \rho e^{i\theta} \), we replace the path of integration in the last two integrals by ones at right-angles to them. Thus
\[ f(s) = 4\left[2\beta\gamma - \beta' + \beta \log (4\pi^2)\right] \frac{\nu^{1-s}}{s - 1} + 4\beta \log \nu \frac{\nu^{1-s}}{s - 1} + 4\beta \frac{\nu^{1-s}}{(s - 1)^2} + 2 \int_{r}^{r-i\infty} \frac{\mu(y)}{y^s} \, dy + 2 \int_{r}^{r+i\infty} \frac{\mu(e^{-i\pi y})}{y} \, dy. \]

But the above integrals define a function of \( s \) which is regular for all finite values of \( s \). Therefore \( f(s) \) is analytic and single valued in the whole \( s \)-plane except perhaps at \( s = 1 \), where it may have a pole of the second order, with the principal part

\[
\frac{8\beta\gamma - 4\beta' + 4\beta \log (4\pi^2)}{s - 1} + \frac{4\beta}{(s - 1)^2}.
\]

Likewise, using (III'), which is a consequence of (b), we can show that \( g(s) \) has the same properties as \( f(s) \), and that the principal part of \( g(s) \) at \( s = 1 \) is

\[
\frac{4\alpha}{(s - 1)^2} + \frac{8\alpha\gamma - 4\alpha' + 4\alpha \log (4\pi^2)}{s - 1}.
\]

Now if we proceed to the limit as \( \nu \to 0 \) in \( \text{Re } s < 1 \), we obtain from (14)

\[
f(s) = 2 \int_{0}^{i\infty} \frac{\mu(y)}{y^s} \, dy + 2 \int_{0}^{i\infty} \frac{\mu(e^{-i\pi y})}{y^s} \, dy
\]

\[
= 4 \sin\left(\frac{\pi}{2} s\right) \int_{0}^{\infty} \frac{\mu(e^{-i\pi/2 t})}{t^s} \, dt.
\]

This gives the integral representation of \( f(s) \) in the half \( s \)-plane.

If we interchange the order of integration and summation in (15) and use

\[
\int_{0}^{\infty} K_{0}(at^{1/2}) t^{-s} \, dt = a^{2s-2} 2^{1-2s} [\Gamma(1-s)]^2, \quad \text{Re } s < 1; \quad |\text{arg } a| < \frac{\pi}{2},
\]

we obtain

\[
f(s) = \left[\sin\left(\frac{\pi}{2} s\right)\right]^2 [\Gamma(1-s)]^2 2^{2s-2} g(1-s) \pi^{2s-2}, \quad \text{Re } s < -1,
\]

which can be written as

\[
f(s) = \pi^{-s} \left[\Gamma\left(\frac{s}{2}\right)\right]^2 f(s) = \pi^{-1+s} \left[\Gamma\left(\frac{1-s}{2}\right)\right]^2 g(1-s).
\]
and by analytic continuation equality (16) holds for all $s$.

Finally, expanding both sides of (16) around $s = 0$ and $s = 1$ respectively, we find that

$$\alpha = f(0), \quad \alpha' = f'(0), \quad \beta = g(0), \quad \beta' = g'(0).$$

Thus we have proved the following: Under conditions (a) and (b), $(s-1)^2f(s)$ and $(s-1)^2g(s)$ are entire functions; $f(s)$ and $g(s)$ satisfy the functional equation (16) and further $\alpha, \alpha', \beta, \beta'$ are related to these functions as given in (17).

Acknowledgment. Professor F. Oberhettinger suggested this problem and gave every encouragement. To him I wish to express my gratitude.

References


Endicott, New York