ON A BOUNDED INCREASING POWER SERIES

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Recently [1] H. S. Shapiro has shown that if k is a positive integer, $\beta=1/(2k)$ and $\alpha=1-\beta$, and if $a_0=0$, $a_n=n^{-\alpha}\cos n^{\beta}$ for $n=1,2,3,\cdots$, then the function

$$f(x)=\sum_0^\infty a_n x^n \tag{1}$$

has bounded variation on $[0, 1)$ but the series

$$\sum_0^\infty a_n \tag{2}$$

is divergent. Given any $\epsilon>0$ we may by choosing k large enough ensure that

$$a_n = O(n^{-1+\epsilon}). \tag{3}$$

However the series (2) is certainly Abel summable and so the stronger condition

$$a_n = O(n^{-1}) \tag{4}$$

would imply the convergence of (2), by Littlewood's Tauberian theorem. Thus Shapiro's example shows that we cannot weaken (4) to a condition of the form (3) in Littlewood's theorem by making the compensating assumption that f has bounded variation on $[0, 1)$.

In this note we prove a stronger negative result than that of Shapiro.

THEOREM. *Let $\phi(n)$ be positive for all positive integers n, and let $\phi(n)\uparrow\infty$ as $n\to\infty$. Then there is a function f of the form* (1) *which is increasing and bounded on* $[0, 1)$ *and for which* (2) *is divergent although*

$$|a_n| < n^{-1}\phi(n) \quad \textit{for all } n\geqq 1. \tag{5}$$

PROOF. Let $\{n_k\}$ $(k=1, 2, 3, \cdots)$ be an increasing sequence of integers satisfying

$$\phi(n_k) > k^2 \qquad (k\geqq 1). \tag{6}$$

Put

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$$(7)\qquad a_n = \begin{cases} k^2/n & \text{if} \qquad k^2 n_k \leqq n < (k^2+1)n_k, \quad k = 1, 2, 3, \cdots \\ -k^2/n & \text{if } (k^2+1)n_k \leqq n < (k^2+2)n_k, \quad k = 1, 2, 3, \cdots \\ 0 & \text{for all other values of } n. \end{cases}$$

(7) defines a_n uniquely for all $n \geqq 0$; and if the summation is over the range $k^2 n_k \leqq n < (k^2+1)n_k$, then

$$\sum a_n = k^2 \sum n^{-1} > k^2 n_k / \{(k^2+1)n_k\} \to 1,$$

so that (2) is divergent. Moreover for each value of n, either $a_n = 0$ or else by (7) there is a positive integer k such that

$$n \mid a_n \mid = k^2, \qquad k^2 n_k \leqq n,$$

and so by (6) and the fact that ϕ is increasing,

$$n \mid a_n \mid < \phi(n_k) \leqq \phi(n),$$

which proves (5).

It remains to show that f, defined by (1), is increasing and bounded on $[0, 1)$. In fact, by (7), $f(x) = \sum_1^\infty f_k(x)$, where

$$f_k(x) = k^2 \sum\nolimits_1 x^n/n - k^2 \sum\nolimits_2 x^n/n,$$

$\sum_1$ being taken over the range $k^2 n_k \leqq n < (k^2+1)n_k$, and $\sum_2$ over the range $(k^2+1)n_k \leqq n < (k^2+2)n_k$. It is easy to see that $f_k'(x) \geqq 0$ for all x on $[0, 1)$, and so $f'(x) \geqq 0$, whence f is increasing on $[0, 1)$. Also, when $x = 1$, $\sum_1$ consists of n_k terms, each at most as big as $1/(k^2 n_k)$, and $\sum_2$ consists of n_k terms, each at least as big as $1/\{(k^2+2)n_k\}$, and so

$$f_k(1) \leqq 1 - k^2/(k^2+2) \sim 2/k^2.$$

Therefore $f(1-0) = \sum_1^\infty f_k(1) < +\infty$. Thus f is increasing and bounded on $[0, 1)$, and this proves the theorem.

Reference

1. H. S. Shapiro, *A remark concerning Littlewood's Tauberian theorem*, Proc. Amer. Math. Soc. **16** (1965), 258–259.

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