A COMPARISON THEOREM FOR ELLIPTIC DIFFERENTIAL EQUATIONS

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Clark and the author [2] recently obtained a generalization of the Hartman-Wintner comparison theorem [4] for a pair of self-adjoint second order linear elliptic differential equations. The purpose of this note is to extend this generalization to general second order linear elliptic equations. As in [2], the usual pointwise inequalities for the coefficients are replaced by a more general integral inequality. The result is new even in the one-dimensional case, and extends Leighton's result for self-adjoint ordinary equations [5].

Protter [6] obtained pointwise inequalities in the nonself-adjoint case in two dimensions by the method of Hartman and Wintner [4]. We obtain an alternative to Protter's result as a corollary of our main theorem.

Let R be a bounded domain in n-dimensional Euclidean space with boundary B having a piecewise continuous unit normal. The linear elliptic differential operator L defined by

$$(1)\qquad Lu = \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + 2\sum_{i=1}^{n} b_iD_iu + cu, \qquad a_{ij} = a_{ji}$$

will be considered in R, where D_i denotes partial differentiation with respect to x^i, $i=1, 2, \cdots, n$. We assume that the coefficients a_{ij}, b_i, and c are real and continuous on $\overline{R}$, the b_i are differentiable in R, and that the symmetric matrix (a_{ij}) is positive definite in R. A "solution" u of $Lu=0$ is supposed to be continuous on $\overline{R}$ and have uniformly continuous first partial derivatives in R, and all partial derivatives involved in (1) are supposed to exist, be continuous, and satisfy $Lu=0$ in R.

Let $Q[z]$ be the quadratic form in $(n+1)$ variables $z_1, z_2, \cdots, z_{n+1}$ defined by

$$(2)\qquad Q[z] = \sum_{i,j=1}^{n} a_{ij}z_iz_j - 2z_{n+1}\sum_{i=1}^{n} b_iz_i + gz_{n+1}^2,$$

where the continuous function g is to be determined so that this form is positive semidefinite. The matrix Q associated with $Q[z]$ has the block form

Received by the editors October 21, 1965.

[1] This research was supported by the United States Air Force Office of Scientific Research, under grant AF-AFOSR-379-65.

$$Q = \begin{pmatrix} A & -b \\ -b^T & g \end{pmatrix}, \qquad A = (a_{ij}),$$

where b^T is the n-vector $(b_1, b_2, \cdots, b_n)$. Let B_i denote the cofactor of $-b_i$ in Q. Since A is positive definite, a necessary and sufficient condition for Q to be positive semidefinite is $\det Q \geqq 0$, or

$$g \det(a_{ij}) \geqq -\sum_{i=1}^{n} b_i B_i. \tag{3}$$

The proof is a slight modification of the well-known proof for positive definite matrices [**3**, p. 306].

Let J be the quadratic functional defined by

$$J[u] = \int_R F[u]\, dx \tag{4}$$

where

$$F[u] = \sum_{i,j} a_{ij} D_i u D_j u - 2u \sum_i b_i D_i u + (g - c) u^2,$$

with domain $\mathfrak{D}$ consisting of all real-valued continuous functions on $\bar{R}$ which have uniformly continuous first partial derivatives in R and vanish on B.

LEMMA. *Suppose g satisfies* (3). *If there exists $u \in \mathfrak{D}$ not identically zero such that $J[u] < 0$, then every solution v of $Lv = 0$ vanishes at some point of $\bar{R}$.*

PROOF. Suppose to the contrary that there exists a solution $v \neq 0$ in $\bar{R}$. For $u \in \mathfrak{D}$ define

$$X^i = v D_i(u/v);$$

$$Y^i = v^{-1} \sum_j a_{ij} D_j v, \qquad i = 1, 2, \cdots, n;$$

$$E[u, v] = \sum_{i,j} a_{ij} X^i X^j - 2u \sum_i b_i X^i + g u^2 + \sum_i D_i(u^2 Y^i).$$

A routine calculation yields the identity

$$E[u, v] = F[u] + u^2 v^{-1} L v.$$

Since $Lv = 0$ in R,

$$J[u] = \int_R \left[\sum_{i,j} a_{ij}X^iX^j - 2u \sum_i b_iX^i + gu^2 \right] dx \tag{5}$$
$$+ \int_R \sum_i D_i(u^2Y^i)\, dx.$$

Since $u=0$ on B, the second integral is zero by Green's formula. The first integrand is a positive semidefinite form by hypothesis (3). The contradiction $J[u] \geqq 0$ establishes the lemma.

Consider in addition to (1) a second differential operator L^* of the same form,

$$\overset{*}{L}u = \sum_{i,j=1}^{n} D_i(\overset{*}{a}_{ij}D_ju) + 2\sum_i \overset{*}{b}_i D_iu + \overset{*}{c}u, \qquad \overset{*}{a}_{ij} = \overset{*}{a}_{ji}$$

in which the coefficients satisfy the same conditions as the coefficients in (1). L^* is the Euler-Jacobi operator associated with the quadratic functional J^* defined by

$$\overset{*}{J}[u] = \int_R \left[\sum_{i,j} \overset{*}{a}_{ij}D_iuD_ju - 2u\sum_i \overset{*}{b}_i D_iu - c^*u^2 \right] dx. \tag{6}$$

Define $V[u] = J^*[u] - J[u]$, $u \in \mathfrak{D}$. Since $u=0$ on B, it follows from partial integration that

$$V[u] = \int_R [\sum (\overset{*}{a}_{ij} - a_{ij})D_iuD_ju \tag{7}$$
$$+ \{ \sum D_i(b_i^* - b_i) + c - c^* - g\} u^2]\, dx.$$

THEOREM 1. *Suppose g satisfies* (3). *If there exists a nontrivial solution u of* $L^*u=0$ *in R such that* $u=0$ *on B and* $V[u]>0$, *then every solution of* $Lv=0$ *vanishes at some point of* $\overline{R}$.

PROOF. The hypothesis $V[u]>0$ is equivalent to $J[u]<J^*[u]$. Since $u=0$ on B, it follows from Green's formula that $J^*[u]=0$. Hence the hypothesis $J[u]<0$ of the lemma is fulfilled.

THEOREM 2. *Suppose* g det $(a_{ij}) > -\sum b_iB_i$. *If there exists a nontrivial solution of* $L^*u=0$ *in R such that* $u=0$ *on B and* $V[u]\geqq 0$, *then every solution of* $Lv=0$ *vanishes at some point of* $\overline{R}$.

Since Q is positive definite, the lemma is valid when the hypothesis $J[u]<0$ is replaced by $J[u]\leqq 0$. The proof of Theorem 2 is then analogous to that of Theorem 1.

In the case that equality holds in (3), that is

$$g = -\sum_i b_i B_i/\det(a_{ij}), \tag{8}$$

define

$$\delta = \sum D_i(b_i^* - b_i) + c - c^* - g.$$

L is called a "strict Sturmian majorant" of L^* by Hartman and Wintner [4] when the following conditions hold: (i) $(a_{ij}^* - a_{ij})$ is positive semidefinite and $\delta \geqq 0$ in $\bar{R}$; (ii) either $\delta > 0$ at some point or $(a_{ij}^* - a_{ij})$ is positive definite and $c^* \neq 0$ at some point. The corollary below follows immediately from Theorem 1.

COROLLARY. *Suppose that L is a strict Sturmian majorant of L^*. If there exists a solution u of $L^*u = 0$ in R such that $u = 0$ on B and u does not vanish in any open set contained in R, then every solution of $Lv = 0$ vanishes at some point of $\bar{R}$.*

If the coefficients a_{ij}^* are of class $C^{2,1}(R)$ (i.e. all second derivatives exist and are Lipschitzian), the hypothesis that u does not vanish in any open set of R can be replaced by the condition that u does not vanish identically in R because of Aronszajn's unique continuation theorem [1].

In the case $n = 2$ considered by Protter [6], the condition $\delta \geqq 0$ reduces to

$$(a_{11}a_{22} - a_{12}^2)\left(\sum_{i=1}^{2} D_i(b_i^* - b_i) + c - c^*\right) \geqq a_{11}b_2^2 - 2a_{12}b_1b_2 + a_{22}b_1^2, \tag{9}$$

which is considerably simpler than Protter's condition. It reduces to Protter's condition

$$\sum_{i=1}^{2} D_i b_i^* + c - c^* \geqq 0$$

in the case that $b_1 = b_2 = 0$, and also in the case that $a_{12} = a_{12}^* = 0$, $a_{11} = a_{11}^*$, $a_{22} = a_{22}^*$. (Two incorrect signs appear in [6]).

The following example in the case $n = 2$ illustrates that Theorem 1 is more general than the pointwise condition (9). Let R be the square $0 < x^1, x^2 < \pi$. Let L^*, L be the elliptic operators defined by

$$L^*u = D_1^2 u + D_2^2 u + 2u,$$
$$Lv = D_1^2 v + D_2^2 v + D_1 v + cv,$$

where

$$c(x^1, x^2) = f(x^1)f(x^2) + 5/4,$$

and $f \in C[0, \pi]$ is not identically zero. The function $u = \sin x^1 \sin x^2$ is zero on B and satisfies $L^*u = 0$. The condition $V[u] > 0$ of Theorem 1 reduces to

$$\int_0^\pi \int_0^\pi f(x^1)f(x^2) \sin^2 x^1 \sin^2 x^2 \, dx^1 dx^2 > 0.$$

Since this is fulfilled, every solution of $Lv = 0$ vanishes at some point of $\bar{R}$. This cannot be concluded from (9) or from Protter's result [**6**] unless f has constant sign in R.

In the case $n = 1$, L is an ordinary differential operator of the form

$$Lu = (au')' + 2bu' + cu,$$

and R is an interval (x_1, x_2). We assert that $\bar{R}$ can be replaced by R in the lemma and theorems; for v can have at most a simple zero at the boundary points x_1 and x_2, and hence the first integral on the right side of (5) exists and is nonnegative provided only that $v \neq 0$ in R.

THEOREM 3. *If there exists a nontrivial solution u of $L^*u = 0$ in R such that $u = 0$ on B and*

$$\text{(10)} \qquad \int_{x_1}^{x_2} \left[(a^* - a)u'^2 + \left(b^{*\prime} - b' + c - c^* - \frac{b^2}{a} \right) u^2 \right] dx > 0,$$

then every solution of $Lv = 0$ has a zero in (x_1, x_2).

In the self-adjoint case $b = b^* = 0$ it was shown by Clark and the author [**2**] that the strict inequality in the hypothesis $V[u] > 0$ of Theorem 1, and therefore also in (10), can be replaced by $\geqq$. Indeed, this is transparent when the proof of the above lemma is specialized to the self-adjoint case. With $>$ replaced by $\geqq$, (10) reduces to Leighton's condition in the self-adjoint case [**5**].

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q-ANALOGUES OF CAUCHY'S FORMULAS

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1. Let q be a given number and let α be real or complex. The αth "basic number" is defined by means of $[\alpha]=(1-q^\alpha)/(1-q)$. This has served as a basis for an extensive amount of literature in mathematics under such titles as Heine, basic, or q-series and functions. The basic numbers also occur naturally in many theta identities. The works of Jackson (for bibliography see [2]) and Hahn [3] have stimulated much interest in this field.

One important operation that is intimately connected with basic series as well as with difference and other functional equations is the q-derivative of a function f. This is defined by

$$Df(x) = \frac{f(qx) - f(x)}{x(q-1)}\,. \tag{1.1}$$

Jackson defined the operations, which he called q-integration,

$$\int_0^x f(t)d(q,t) = x(1-q)\sum_{k=0}^{\infty} q^k f(xq^k) \tag{1.2}$$

and

$$\int_x^\infty f(t)d(q,t) = x(1-q)\sum_{k=0}^{\infty} q^{-k} f(xq^{-k}) \tag{1.3}$$

Received by the editors October 28, 1965 and, in revised form, November 30, 1965.