ON THE GENERALISATION OF A FORMULA OF RAINVILLE

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1. **Introduction.** Rainville [8] obtained a formula for $P_n(\text{Cos }\alpha)$ where $P_n(x)$ is the Legendre polynomial. This was later generalised by Carlitz [5] who obtained a formula for $C_n^{\lambda+1/2}$ (Cos α), where $C_n^{\lambda}(x)$ is the Ultraspherical polynomial. Banerjee [1], Yadao [11] and Rangarajan [10] obtained similar formulae for the associated Legendre function $P_n^m(x)$.

The object of this short paper is to obtain a more general formula from which the above formulae follow as particular cases.

2. **The generalised formula.** Let $E(u)$ be the exponential function and let $G(u)$ possess a power series expansion (convergent or divergent)

$$(2.1)\qquad G(u) = \sum_{n=0}^{\infty} g_n u^n, \qquad g_n \neq 0.$$

Define the sequence of polynomials $F_n(x)$ by the generating relation (essentially the one used in Example 21, p. 186, of Rainville [9])

$$(2.2)\qquad E(xt)G(\tfrac{1}{4}t^2(x^2-1)) = \sum_{n=0}^{\infty} \frac{F_n(x)t^n}{n!}.$$

It follows, by multiplication of power series and equating coefficients of t^n, that

$$(2.3)\qquad F_n(x) = \sum_{k=0}^{[n/2]} \frac{n!g_k x^{n-2k}(x^2-1)^k}{2^{2k}(n-2k)!}.$$

The most useful special cases seem to occur when $G(u)$ is of hypergeometric form,

$$(2.4)\qquad G(u) = {}_pF_q\left[\begin{matrix} a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix}\; u\right]$$

which leads to the polynomial set

$$(2.5)\qquad F_n(x) = x^n\,{}_{p+2}F_q\left[\begin{matrix} -n/2,\ -(n-1)/2,\ a_1, \cdots, a_p; \\ b_1, \cdots, b_q; \end{matrix}\; \frac{x^2-1}{x^2}\right].$$

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Putting $t=v(y^2-1)^{1/2}$ in (2.2) we get

$$(2.6)\quad E[vx(y^2-1)^{1/2}]G[\tfrac{1}{4}v^2(x^2-1)(y^2-1)] = \sum_{n=0}^{\infty} \frac{v^n(y^2-1)^{n/2}}{n!} F_n(x).$$

Interchanging x and y,

$$(2.7)\quad E[vy(x^2-1)^{1/2}]G[\tfrac{1}{4}v^2(x^2-1)(y^2-1)] = \sum_{n=0}^{\infty} \frac{v^n(x^2-1)^{n/2}}{n!} F_n(y).$$

Dividing (2.6) by (2.7) we have

$$(2.8)\quad \begin{aligned} &\sum_{n=0}^{\infty} \frac{v^n(y^2-1)^{n/2}}{n!} F_n(x) \\ &\quad = E\{v(x(y^2-1)^{1/2} - y(x^2-1)^{1/2})\} \sum_{n=0}^{\infty} \frac{v^n(x^2-1)^{n/2}}{n!} F_n(y). \end{aligned}$$

Now equating coefficients of v^n from both sides, we obtain

$$(2.9)\quad F_n(x) = \left(\frac{1-x^2}{1-y^2}\right)^{n/2} \sum_{k=0}^{n} \binom{n}{k} \left[\frac{x(1-y^2)^{1/2}-y(1-x^2)^{1/2}}{(1-x^2)^{1/2}}\right]^{n-k} F_k(y).$$

This can also be written as

$$(2.10)\quad F_n(x) = \left(\frac{1-x^2}{1-y^2}\right)^{n/2} \sum_{k=0}^{n} \binom{n}{k} \left[\frac{x(1-y^2)^{1/2}-y(1-x^2)^{1/2}}{(1-x^2)^{1/2}}\right]^{k} F_{n-k}(y).$$

Here $F_n(x)$ is expressed as the sum of a series of $F_n(y)$.

3. **Special cases of the above formula.** Putting $x=\cos\alpha$, $y=\cos\beta$ in (2.9) we get,

$$(3.1)\quad F_n(\cos\alpha) = \left(\frac{\sin\alpha}{\sin\beta}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left[\frac{\sin(\beta-\alpha)}{\sin\alpha}\right]^{n-k} F_k(\cos\beta).$$

Putting $x=\sin\alpha$, $y=\cos\beta$ in (2.9) or changing α into $\pi/2-\alpha$ in (3.1),

$$(3.2)\quad F_n(\sin\alpha) = \left(\frac{\cos\alpha}{\sin\beta}\right)^n \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \left[\frac{\cos(\alpha+\beta)}{\cos\alpha}\right]^{n-k} F_k(\cos\beta).$$

Putting $\beta=2\alpha$ in (3.1),

$$(3.3)\quad (2\cos\alpha)^n F_n(\cos\alpha) = \sum_{k=0}^{n} \binom{n}{k} F_k(\cos 2\alpha).$$

Putting $x=\sin\beta$, $y=-\cos\beta$ in (2.9) or changing α into $\pi/2+\beta$ in (3.3) we get

$$(3.4)\qquad \sin^n \beta F_n(\sin\beta) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \cos^k \beta F_k(\cos\beta).$$

Putting $y=-x$ in (2.9) we get

$$(3.5)\qquad F_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2x)^{n-k} F_k(x).$$

Putting $x=\cosh\alpha$, $y=\cosh\beta$ in (2.9) or changing α into $i\alpha$ and β into $i\beta$ in (3.1), we get

$$(3.6)\quad F_n(\cosh\alpha) = \left(\frac{\sinh\alpha}{\sinh\beta}\right)^n \sum_{k=0}^{n} \binom{n}{k} \left[\frac{\sinh(\beta-\alpha)}{\sinh\alpha}\right]^{n-k} F_k(\cosh\beta).$$

Putting $x=\cos 2\theta$, $y=\cos\theta$ in (2.9)

$$(3.7)\qquad F_n(\cos 2\theta) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (2\cos\theta)^k F_k(\cos\theta).$$

Putting $\cos\theta = x$ in (3.7)

$$(3.8)\qquad F_n(1-2x^2) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (2x)^k F_k(x).$$

Changing x to $((1+x)/2)^{1/2}$ and y to x, we get from (2.9)

$$(3.9)\qquad 2^{n/2}(1+x)^{n/2} F_n\left(\left(\frac{1+x}{2}\right)^{1/2}\right) = \sum_{k=0}^{n} \binom{n}{k} F_k(x).$$

Putting $(1-xt)/\rho$ for x and $-x$ for y in (2.9), where $\rho=(1-2xt+t^2)^{-1/2}$ we get

$$(3.10)\qquad \rho^n F_n\left(\frac{1-xt}{\rho}\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^k F_k(x).$$

Putting $(x-t)/\rho$ for x and x for y in (2.10), we have

$$(3.11)\qquad \rho^n F_n\left(\frac{x-t}{\rho}\right) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} t^k F_{n-k}(x).$$

In this way by giving different values to x and y, we can obtain various relations of similar types.

4. **Particular cases.** Putting $p=0$, $q=1$ and $b_1=1$ in (2.5), we see that $F_n(x)$ becomes the Legendre polynomial $P_n(x)$. Hence the formula (8) of Rainville [8] and its special cases become particular cases of (3.1), (3.3), (3.4), (3.5) and the result (3.9) above now re-

duces to the result given by Bhonsle [2]. The results of Example 7 and Example 9, p. 184 of Rainville [9] follow easily from (3.4), (3.5), (3.8) and (3.10) respectively.

Putting $p=0$, $q=1$, $b_1=\lambda+\frac{1}{2}$ in (2.5) we find that $F_n(x)$ becomes

$$\frac{n!}{(2\lambda)_n} C_n^\lambda(x),$$

where $C_n^\lambda(x)$ is the Ultraspherical polynomial defined by $(1-2xt+t^2)^{-\lambda} = \sum C_n^\lambda(x)t^n$. The relation (2.9) then transforms into

$$C_n^\lambda(x) = \left(\frac{1-x^2}{1-y^2}\right)^{n/2} \cdot \sum_{k=0}^{n} \frac{(2\lambda)_n}{(n-k)!(2\lambda)_k} \left[\frac{x(1-y^2)^{1/2}-y(1-x^2)^{1/2}}{(1-x^2)^{1/2}}\right]^{n-k} C_k^\lambda(y). \tag{4.1}$$

With the substitution $y=2x^2-1$, (4.1) now reduces to the result (4.4) of Chatterjea [6]. The result given by Carlitz [5] is also a particular case of (4.1).

Denoting $\Phi_n(x)$ (as given by Rainville [8]) by the relation

$$\Phi_n(x) = (1-x^2)^{n/2} P_n\left(\frac{1}{(1-x^2)^{1/2}}\right) \tag{4.2}$$

(2.9) reduces to

$$y^n\Phi_n(x) = \sum_{k=0}^{n} \binom{n}{k} (y-x)^{n-k} x^k \Phi_k(y). \tag{4.3}$$

The results given by Chatterjea [6], [7] now follow easily from (4.3).

Putting $p=0$, $q=1$, $b_1=m+1$ in (2.5), we obtain

$$F_n(x) = (x^2-1)^{-m/2} \frac{2^m m!n!}{(2m)!(2m+1)_n} P_{m+n}^m(x)$$

and then (2.9) takes the form

$$P_{m+n}^m(x) = \left(\frac{1-x^2}{1-y^2}\right)^{(m+n)/2} \cdot \sum_{k=0}^{n} \binom{2m+n}{k} \left(\frac{x(1-y^2)^{1/2}-y(1-x^2)^{1/2}}{(1-x^2)^{1/2}}\right)^k P_{m+n-k}^m(y). \tag{4.4}$$

The results (5), (6), (7) of Banerjee [1] and the result (2.3) of Rangarajan [10] now follow with proper substitutions from (4.4) as in §3.

From (3.10), putting $p=0$, $q=1$, $b_1=\alpha+1$, we get

$$(4.5)\qquad \rho^n \frac{n!}{(1+\alpha)_n} P_n^{(\alpha,\alpha)}\left(\frac{1-xt}{\rho}\right) = \sum_{k=0}^{n} \frac{(-n)_k P_k^{(\alpha,\alpha)}(x) t^k}{(1+\alpha)_k}$$

which has been obtained by Brafman [4] by a different method. Similarly from (3.11) we obtain

$$(4.6)\qquad \rho^n C_n^{\nu}\left(\frac{x-t}{\rho}\right) = \sum_{k=0}^{n} (-1)^k \frac{t^k}{k!} \frac{\Gamma(n+2\nu)}{\Gamma(n-k+2\nu)} C_{n-k}^{\nu}(x).$$

Incidentally it may be mentioned that the result of Bloh [3]

$$E(tz)\, I_m\,(t(z^2-1)^{1/2}) = \sum_{n=0}^{\infty} \frac{t^{m+n} P_{m+n}^{m}(z)}{(2m+n)!}$$

from which the results of Banerjee [1] were deduced, can be easily obtained from (2.2) with the substitutions $p=0$, $q=1$, $b_1=1+m$ and the definitions of $P_{m+n}^m(x)$ and $I_m(x)$.

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