SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is no other outlet.

OPERATORS WITH CLOSED RANGE[1]

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Part C of this note is Banach's Theorem [1] on operators with closed range. It differs from the proofs known to us in avoiding use of the w^* topology.[2]

A. Let T be a bounded linear operator from a Banach space X to a normed linear space Y, $\epsilon>0$, $0\leqq\rho<1$. Then T is an open map of X onto Y if, for each $y\in Y$, there exists $x\in X$ such that

$$\epsilon\|x\| + \|y - Tx\| \leqq \rho\|y\|. \tag{1}$$

PROOF. Let $y_0\in Y$, and choose a sequence $x_1, x_2, x_3, \cdots$ in Y such that $\epsilon\|x_1\|+\|y_0-Tx_1\|\leqq\rho\|y_0\|$ and for $n>1$,

$$\epsilon\|x_n\| + \|y_0 - Tx_1 - \cdots - Tx_{n-1} - Tx_n\|$$
$$\leqq \rho\|y_0 - Tx_1 - \cdots - Tx_{n-1}\|.$$

These inequalities yield further $\|y_0-Tx_1- \cdots -Tx_n\|\leqq\rho^n\|y_0\|$ for $n\geqq 1$ and $\epsilon\|x_n\|\leqq\rho^n\|y_0\|$. Let s be the absolutely convergent sum $\sum x_n$, so that $\epsilon\|s\|\leqq\rho(1-\rho)^{-1}\|y_0\|$ and $Ts=y_0$. This proves A.

B. If $y_0\in Y$ and equation (1) is *false* for every x, there is a function f in Y^* such that $\rho\leqq\|f\|\leqq 1$ and $\|T^*f\|\leqq\epsilon$. (Cf. [2].)

PROOF. Let $Z=X+Y$, with $\|(x, y)\|=\|x\|+\|y\|$ and $V=\{(\epsilon x, Tx): x\in X\}$. Then $\|\epsilon x\|+\|y_0-Tx_1\|>\rho\|y_0\|$ for every x, so there is a linear functional F in Z^* such that $\|F\|=1$, $F(V)=0$, and $|F(0, y_0)|\geqq\rho\|y_0\|$. Take $f(y)\equiv F(0, y)$, $y\in Y$. Then $\rho\leqq\|f\|\leqq 1$, while for each x in X, $T^*f(x)=f(Tx)=F(0, Tx)=F(-\epsilon x, 0)=-\epsilon F(x, 0)$ yielding $\|T^*f\|\leqq\epsilon$.

C. If TX is not closed in Y, T^*Y^* is not closed in X^*.

PROOF. If W denotes the closure of TX in Y, the Hahn-Banach theorem shows that the range T^*Y^* can be identified with the range

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[2] *Added in proof.* This does not apply to the proof given by Kahane and Salem, *Ensembles parfaits,* which uses an argument like A. The reader should certainly refer to this also.

T^*W^*. If, then, $TX \neq W$, there is a sequence w_n^* in W^* such that $\|w_n^*\| \to 1$, while $T^*w_n^* \to 0$. T^* being 1-1 on W^*, it cannot be an open mapping onto T^*W^*, whence the last subspace is not closed.

References

1. S. Banach, *Théorie des operations linéaires*, Monografie Matematyczne, Warsaw, 1932; pp. 145–152.

2. N. Dunford and J. Schwartz, *Linear operators*. I, Interscience, New York, 1958; Lemma 3, p. 488.

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A SHORT PROOF OF JACOBI'S FOUR SQUARE THEOREM

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Let $R_4(n)$ denote the number of representations of n as a sum of four squares and let $R_4'(4m)$, where m is odd, denote the number of representations of $4m$ as a sum of four odd squares. It is familiar that

$$R_4'(4m) = 16\sigma(m) \tag{1}$$

and

$$R_4(n) = \begin{cases} 8\sigma'(n) & (n \text{ odd}), \\ 24\sigma'(n) & (n \text{ even}), \end{cases} \tag{2}$$

where

$$\sigma(n) = \sum_{d|n} d, \qquad \sigma'(n) = \sum_{d|n;\, d \text{ odd}} d.$$

These results can be proved rapidly as follows. In the usual notation of elliptic functions put [2, Chapter 21]

$$\lambda = k^2 = \frac{\theta_2^4}{\theta_3^4}, \qquad 1 - \lambda = \frac{\theta_0^4}{\theta^4}\,.$$

Then

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