A DYNAMIC PROGRAMMING GENERALIZATION OF xy TO n VARIABLES

A. P. HILLMAN, D. G. MEAD, K. B. O'KEEFE[1] AND E. S. O'KEEFE

A function $\phi(d_1, \cdots, d_n)$, from the ordered n-tuples of nonnegative integers into the nonnegative integers augmented by ∞, with the following properties arises in the investigation [6] of the differential ideal generated by a product of n differential indeterminates:

(A) $\phi(d_1, \cdots, d_n)$ is symmetric in the d_i.

(B) $\phi(0)=0$ and $\phi(x)=\infty$ for $x>0$.

(C) $\phi(x, y)=xy$.

(D) $\phi(d_1, \cdots, d_n)$

$$=\min[\phi(d_1-r, \cdots, d_a-r)+\phi(d_{a+1}-s, \cdots, d_b-s)+\cdots +\phi(d_{c+1}-t, \cdots, d_n-t)+\phi(r, s, \cdots, t)]$$

where $r, s, \cdots, t$ range over all nonnegative integers such that the arguments are nonnegative.

Many other properties of ϕ can be derived from these four, including the following:

(E) $\phi(d_1, \cdots, d_n)=\min[\phi(d_1-t, \cdots, d_{n-1}-t)+td_n]$.

$$\text{(F)}\ \phi(d_1, \cdots, d_n)=\min\left[(d_{n-1}-T_{n-2})(d_n-T_{n-2})+\sum_{i=1}^{n-2} t_i(d_i-T_{i-1})\right],$$

where $T_i=t_1+\cdots+t_i$.

Property (F) shows that ϕ is a solution of a quadratic integer programming problem. Some references are given at the close of this paper; others may be found in these listed publications.

We assume using (A) that the d_i are numbered so that $d_1\leqq d_2\leqq \cdots \leqq d_n$ and then use (C) and (E) to find an explicit expression for ϕ.

Let $D=(d_1, \cdots, d_n)$. For $2\leqq i\leqq n$ let $q_i=q_i(D)=(d_1+\cdots+d_i)\cdot(i-1)^{-1}$ and let $k=k(D)$ be the smallest i for which q_i assumes its minimum value. Let integers $q=q(D)$ and $r=r(D)$ be defined by $d_1+\cdots+d_k=q(k-1)+r$ and $0\leqq r<k-1$. Let $c_i=q-d_i$ for $i=1, \cdots, k$ and let $C=C(D)=(c_1, \cdots, c_k; r)$. Let $s_1=c_1+\cdots+c_k$ and $s_2=\sum_{i<j} c_ic_j$. We define a function $f(D)$ by

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$$f(D) = s_2 + rs_1 + [(r+1)r/2]$$

and show by induction on n that $f(D)=\phi(D)$, i.e., $f(D)$ satisfies (E) and (C).

If $d_1, \cdots, d_n$ are allowed to take on all real values, it can be shown easily by induction on n that the function $\phi(D)$ satisfying (C) and (E), with t ranging over the real numbers, is

$$\phi(D) = \sum_{i<j} d_i d_j - \frac{n-2}{2(n-1)} (d_1 + \cdots + d_n)^2.$$

Going back to the integer case, we note that $s_1=(q-d_1)+\cdots+(q-d_k)=kq-[(k-1)q+r]=q-r$ and hence that

(1) $$q(D) = s_1(D) + r(D).$$

We also observe that q_i is the average of the $i-1$ integers d_1+d_2, $d_3, \cdots, d_i$. Therefore one and only one of the following holds for each $i \geqq 3$:

(2) (a) $q_{i-1} < q_i < d_i$, (b) $q_{i-1} = q_i = d_i$, (c) $q_{i-1} > q_i > d_i$.

Since the d's are nondecreasing, if $q_j \leqq q_{j+1}$ for some j then (2) implies that $q_j \leqq q_{j+1} \leqq q_{j+2} \leqq \cdots \leqq q_n$. The first such j must therefore be k. Hence we have

(3) $q_2 > q_3 > \cdots > q_k$ and $q_k \leqq q_{k+1} \leqq \cdots \leqq q_n$.

It follows from (2) and (3) that

(4) $d_i < q_i \leqq q_{i+1} \leqq d_{i+1}$ if and only if $i = k$.

Since $(k-1)q+r=d_1+\cdots+d_k=(k-1)q_k$ and $0 \leqq r < k-1$, we also note that q is the greatest integer $[q_k]$ in q_k.

In the case $n=2$, we always have $k=2$, $q=d_1+d_2$, $r=0$, $C=(d_2, d_1; 0)$ and $f(d_1, d_2)=s_2=d_1d_2$. Hence $f(d_1, d_2)=\phi(d_1, d_2)$. We assume that $f(D)=\phi(D)$ for $n=m$ and show that it follows for $n=m+1$.

When $n=m+1$, we let $d_{m+1}=d$, $D^*=(d_1, \cdots, d_m, d)$, and $D=(d_1, \cdots, d_m)$. Using the hypothesis of the induction and (E) we see that $\phi(D^*)$ is

(5) $$\min[f(D-tI)+td]$$

where t ranges over $1, 2, \cdots, d_1$ and $D-tI$ denotes $(d_1-t, \cdots, d_m-t)$.

Let $F(t)=f(D-tI)+td$ and let $\nabla F(t)=F(t)-F(t-1)$. Clearly $\nabla F(t)=d-\Delta f(D-tI)$ where $\Delta f(D)=f(D+I)-f(D)$. We first prove the following:

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LEMMA. *$q(D+I)-q(D)$ is 1 if $r(D+I)>0$ and is 2 if $r(D+I)=0$, $\Delta f(D)=q(D)+1$, and $\nabla F(t)=d-q(D-tI)-1$.*

PROOF. Let $C(D)=(c_1, \cdots, c_k; r)$. When $k(D+I)=k(D)$, it is clear that if $r<k-2$ then $q(D+I)=q(D)+1$ and that $C(D+I)=(c_1, \cdots, c_k; r+1)$ while $q(D+I)=q(D)+2$ and $C(D+I)=(c_1+1, \cdots, c_k+1; 0)$ if $r=k-2$. In each of these cases, one calculates that $f(D+I)=f(D)+s_1(D)+r(D)+1$. This and (1) lead to $\Delta f(D)=q(D)+1$. Then $\nabla F(t)=d-\Delta f(D-tI)=d-q(D-tI)-1$.

Next we note that

$$q_i(D + I) = q_i(D) + 1 + 1/(i - 1). \tag{6}$$

Since $1/(i-1)$ is a decreasing function of i for $i \geqq 2$, this implies that $k(D+I) \geqq k(D)$. Hence the only case remaining is that in which $k(D+I)>k=k(D)$. We then have

$$q_{k+1}(D) \geqq q_k(D), \qquad q_{k+1}(D + I) < q_k(D + I). \tag{7}$$

Using (6) and (7) we obtain

$$\begin{aligned} 0 &< q_k(D + I) - q_{k+1}(D + I) \\ &= \frac{1}{k-1} - \frac{1}{k} - q_{k+1}(D) - q_k(D) \leqq \frac{1}{(k-1)k}. \end{aligned} \tag{8}$$

Since q_{k+1} and q_k are both expressible as integers divided by $(k-1)k$, (8) implies that $q_k(D+I)-q_{k+1}(D+I)=1/(k-1)k$ and

$$q_{k+1}(D) = q_k(D). \tag{9}$$

One sees from (9) and (2) that $d_{k+1}=q_k(D)$. One similarly finds that $d_i=q_k(D)$ for $k<i \leqq k(D+I)$.

Since $q_k(D)=d_{k+1}$ is an integer, we have $q(D)=q_k(D)$ and $r(D)=0$. Hence $d_i=q(D)$ for $k+1 \leqq i \leqq k(D+I)$ and it is easily seen that $q(D+I)=q(D)+1$, $r(D+I)=r(D)+1=1$, and $C(D+I)=(c_1, \cdots, c_k, 0, \cdots, 0; 1)$. One now calculates that $f(D+I)-f(D)=s_1(D)+1=q(D)+1$. As before, $\nabla F(t)=d-q(D-tI)-1$, which completes the proof of the lemma.

It follows from the lemma that $\nabla F(t)$ increases by 1 or 2 and $q(D-tI)$ decreases by 1 or 2 when t increases by 1.

We prove that $f(D^*)=\phi(D^*)$ by considering cases. First let $d \geqq q(D)$. For $1 \leqq t \leqq d_1$, we then have $d \geqq q(D)>q(D-tI)$ and so $\nabla F(t)=d-q(D-tI)-1 \geqq 0$. Hence the minimum $\phi(D^*)$ of the $F(t)$ is $F(0)=f(D)$. If $k(D^*)>k(D)$, we must have $k(D)=m$ and $d \geqq q(D)$ implies that $C(D)$ and $C(D^*)$ are of the form $(c_1, \cdots, c_m; r)$ and

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$(c_1, \cdots, c_m, 0; r)$ respectively. If $k(D^*)=k(D)$, $C(D^*)=C(D)$. Either way, $f(D^*)=f(D)=\phi(D^*)$.

It remains to consider $d<q(D)$. Then $d_m \leqq d<q(D)\leqq q_k(D)$ and (4) implies that $k(D)=m$ and $k(D^*)=m+1$. Now $d_1>0$ since $d_1=0$ would imply $k(D^*)=2$, which is less than $m+1$ for $m\geqq 2$. Then $d\geqq d_2>d_2-d_1=q(D-d_1I)$. Hence $d-q(D-tI)$ must change sign as t goes from 0 to d_1 in one of the two following ways:

First let there be a c with $0<c<d_1$ such that $d=q(D-cI)$. Then $\nabla F(t)\leqq -1$ for $t\leqq c$ and $\nabla F(t)\geqq 0$ for $t>c$. Hence $\phi(D^*)=F(c)=f(D-cI)+cd$. We let $k=k(D-cI)$ and show that $k=m$. Since $c>0$,

$$q_k(D-cI)\leqq d_{k+1}-c<d=q(D-cI)\leqq q_k(D-cI)$$

would be a contradiction unless $k=m$ (and d_{k+1} does not exist). Now it is clear that $q(D^*)=q(D-cI)+c$. Letting $C(D-cI)=(c_1, \cdots, c_m; r)$, one has $C(D^*)=(c_1, \cdots, c_m, c; r)$ and $f(D^*)=f(D-cI)+c(c_1+\cdots+c_m)+cr=f(D-cI)+cd=\phi(D^*)$.

Finally let there be a c with $0\leqq c<d_1$, $d=q[D-(c+1)I]+1$, and $d=q(D-cI)-1$. Then $\nabla F(t)\leqq 0$ for $t<c$ and $\nabla F(t)\geqq 2$ for $t\geqq c$. Hence $\phi(D^*)=F(c)=f(D-cI)+cd$. Since $q(D-cI)-q[D-(c+1)I]=2$, the lemma tells us that $r(D-cI)=0$. As before, $k(D-cI)=m$. Let $C(D-cI)=(c_1, \cdots, c_m; 0)$. One then calculates that $q(D^*)=q(D-cI)+c-1$, $C(D^*)=(c_1-1, \cdots, c_m-1, c; m-1)$, and $f(D^*)=f(D-cI)+(c_1+\cdots+c_m-1)c=f(D-cI)+dc=\phi(D^*)$. This completes the proof.

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University of Santa Clara and
University of Washington, Seattle

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