A DYNAMIC PROGRAMMING GENERALIZATION OF $xy$ TO $n$ VARIABLES

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A function $\phi(d_1, \ldots, d_n)$, from the ordered $n$-tuples of nonnegative integers into the nonnegative integers augmented by $\infty$, with the following properties arises in the investigation [6] of the differential ideal generated by a product of $n$ differential indeterminates:

(A) $\phi(d_1, \ldots, d_n)$ is symmetric in the $d_i$.
(B) $\phi(0) = 0$ and $\phi(x) = \infty$ for $x > 0$.
(C) $\phi(x, y) = xy$.
(D) $\phi(d_1, \ldots, d_n)$

$$= \min \left[ \phi(d_1 - r, \ldots, d_a - r) + \phi(d_{a+1} - s, \ldots, d_b - s) + \ldots \\
+ \phi(d_{c+1} - t, \ldots, d_n - t) + \phi(r, s, \ldots, t) \right]$$

where $r, s, \ldots, t$ range over all nonnegative integers such that the arguments are nonnegative.

Many other properties of $\phi$ can be derived from these four, including the following:

(E) $\phi(d_1, \ldots, d_n) = \min \left[ \phi(d_1 - t, \ldots, d_{n-1} - t) + td_n \right]$.

(F) $\phi(d_1, \ldots, d_n) = \min \left[ (d_{n-1} - T_{n-2})(d_n - T_{n-2}) + \sum_{i=1}^{n-2} t_i(d_i - T_{i-1}) \right]$,

where $T_i = t_1 + \ldots + t_i$.

Property (F) shows that $\phi$ is a solution of a quadratic integer programming problem. Some references are given at the close of this paper; others may be found in these listed publications.

We assume using (A) that the $d_i$ are numbered so that $d_1 \leq d_2 \leq \ldots \leq d_n$ and then use (C) and (E) to find an explicit expression for $\phi$.

Let $D = (d_1, \ldots, d_n)$. For $2 \leq i \leq n$ let $q_i = q_i(D) = (d_1 + \ldots + d_i) \cdot (i - 1)^{-1}$ and let $k = k(D)$ be the smallest $i$ for which $q_i$ assumes its minimum value. Let integers $q = q(D)$ and $r = r(D)$ be defined by $d_1 + \ldots + d_k = q(k - 1) + r$ and $0 \leq r < k - 1$. Let $c_i = q - d_i$ for $i = 1, \ldots, k$ and let $C = C(D) = (c_1, \ldots, c_k, r)$. Let $s_1 = c_1 + \ldots + c_k$ and $s_2 = \sum_{i<j} c_i c_j$. We define a function $f(D)$ by

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$$f(D) = s_2 + rs_1 + [(r + 1)r/2]$$

and show by induction on $n$ that $f(D) = \phi(D)$, i.e., $f(D)$ satisfies (E) and (C).

If $d_1, \ldots, d_n$ are allowed to take on all real values, it can be shown easily by induction on $n$ that the function $\phi(D)$ satisfying (C) and (E), with $t$ ranging over the real numbers, is

$$\phi(D) = \sum_{i<j} d_id_j - \frac{n-2}{2(n-1)} (d_1 + \cdots + d_n)^2.$$

Going back to the integer case, we note that $s_1 = (q-d_1) + \cdots + (q-d_k) = kq - [(k-1)q+r] = q-r$ and hence that

$$q(D) = s_1(D) + r(D).$$

We also observe that $q_i$ is the average of the $i-1$ integers $d_1 + d_2, \ldots, d_i$. Therefore one and only one of the following holds for each $i \geq 3$:

1. $q_{i-1} < q_i < d_i$,
2. $q_{i-1} = q_i = d_i$,
3. $q_{i-1} > q_i > d_i$.

Since the $d$'s are nondecreasing, if $q_i \leq q_{j+1}$ for some $j$ then (2) implies that $q_i \leq q_{j+1} \leq q_{j+2} \leq \cdots \leq q_n$. The first such $j$ must therefore be $k$. Hence we have

$$q_2 > q_3 > \cdots > q_k \quad \text{and} \quad q_k \leq q_{k+1} \leq \cdots \leq q_n.$$

It follows from (2) and (3) that

$$d_i < q_i \leq q_{i+1} \leq d_{i+1} \quad \text{if and only if } i = k.$$

Since $(k-1)q+r = d_1 + \cdots + d_k = (k-1)q_k$ and $0 \leq r < k-1$, we also note that $q$ is the greatest integer $[q_k]$ in $q_k$.

In the case $n = 2$, we always have $k = 2$, $q = d_1 + d_2$, $r = 0$, $C = (d_2, d_1; 0)$ and $f(d_1, d_2) = s_2 = d_1d_2$. Hence $f(d_1, d_2) = \phi(d_1, d_2)$. We assume that $f(D) = \phi(D)$ for $n = m$ and show that it follows for $n = m + 1$.

When $n = m + 1$, we let $d_{m+1} = d$, $D^* = (d_1, \ldots, d_m, d)$, and $D = (d_1, \ldots, d_m)$. Using the hypothesis of the induction and (E) we see that $\phi(D^*)$ is

$$\min[f(D - tI) + td]$$

where $t$ ranges over $1, 2, \ldots, d_1$ and $D - tI$ denotes $(d_1-t, \ldots, d_m-t)$.

Let $F(t) = f(D - tI) + td$ and let $\nabla F(t) = F(t) - F(t-1)$. Clearly $\nabla F(t) = d - \Delta f(D - tI)$ where $\Delta f(D) = f(D+I) - f(D)$. We first prove the following:
Lemma. \( q(D+1) - q(D) \) is 1 if \( r(D+I) > 0 \) and is 2 if \( r(D+I) = 0 \), \( \Delta f(D) = q(D) + 1 \), and \( \nabla F(t) = d - q(D-tI) - 1 \).

Proof. Let \( C(D) = (c_1, \cdots, c_k; r) \). When \( k(D+I) = k(D) \), it is clear that if \( r < k-2 \) then \( q(D+I) = q(D) + 1 \) and that \( C(D+I) = (c_1, \cdots, c_k; r+1) \) while \( q(D+I) = q(D) + 2 \) and \( C(D+I) = (c_1+1, \cdots, c_k+1; 0) \) if \( r = k-2 \). In each of these cases, one calculates that \( f(D+I) = f(D) + s_1(D) + r(D) + 1 \). This and (1) lead to \( \Delta f(D) = q(D) + 1 \). Then \( \nabla F(t) = d - \Delta f(D-tI) = d - q(D-tI) - 1 \).

Next we note that

\[
q_i(D + I) = q_i(D) + 1 + 1/(i - 1).
\]

Since \((i-1)/(i-1)\) is a decreasing function of \(i\) for \(i \geq 2\), this implies that \(k(D+I) \geq k(D)\). Hence the only case remaining is that in which \(k(D+I) > k(k(D))\). We then have

\[
q_{k+1}(D) \geq q_k(D), \quad q_{k+1}(D + I) < q_k(D + I).
\]

Using (6) and (7) we obtain

\[
0 < q_k(D + I) - q_{k+1}(D + I)
\]

\[
= \frac{1}{k-1} - \frac{1}{k} - q_{k+1}(D) - q_k(D) \leq \frac{1}{(k-1)k}.
\]

Since \(q_{k+1}\) and \(q_k\) are both expressible as integers divided by \((k-1)k\), (8) implies that \(q_k(D+I) - q_{k+1}(D+I) = 1/(k-1)k\) and

\[
q_{k+1}(D) = q_k(D).
\]

One sees from (9) and (2) that \(d_{k+1} = q_k(D)\). One similarly finds that \(d_i = q_k(D)\) for \(k < i \leq k(D+I)\).

Since \(q_k(D) = d_{k+1}\) is an integer, we have \(q(D) = q_k(D)\) and \(r(D) = 0\). Hence \(d_i = q(D)\) for \(k+1 \leq i \leq k(D+I)\) and it is easily seen that \(q(D+I) = q(D) + 1\), \(r(D+I) = r(D) + 1 = 1\), and \(C(D+I) = (c_1, \cdots, c_k, 0, \cdots, 0; 1)\). One now calculates that \(f(D+I) - f(D) = s_1(D) + 1 = q(D) + 1\). As before, \(\nabla F(t) = d - q(D-tI) - 1\), which completes the proof of the lemma.

It follows from the lemma that \(\nabla F(t)\) increases by 1 or 2 and \(q(D-tI)\) decreases by 1 or 2 when \(t\) increases by 1.

We prove that \(f(D^*) = \phi(D^*)\) by considering cases. First let \(d \geq q(D)\). For \(1 \leq t \leq d\), we then have \(d \geq q(D) > q(D-tI)\) and so \(\nabla F(t) = d - q(D-tI) - 1 \geq 0\). Hence the minimum \(\phi(D^*)\) of the \(F(t)\) is \(F(0) = f(D)\). If \(k(D^*) > k(D)\), we must have \(k(D) = m\) and \(d \geq q(D)\) implies that \(C(D)\) and \(C(D^*)\) are of the form \((c_1, \cdots, c_m; r)\) and
(c₁, ⋯, cₘ, 0; r) respectively. If k(D*) = k(D), C(D*) = C(D).
Either way, f(D*) = f(D) = ϕ(D*).

It remains to consider d < q(D). Then dₘ ≤ d < q(D) ≤ qₖ(D) and (4)
implies that k(D) = m and k(D*) = m + 1. Now d₁ > 0 since d₁ = 0 would
imply k(D*) = 2, which is less than m + 1 for m ≥ 2. Then d ≥ d₂ > d₃
− d₁ = q(D − d₁I). Hence d − q(D − t₁I) must change sign as t goes from
0 to d₁ in one of the two following ways:

First let there be a c with 0 < c < d₁ such that d = q(D − c₁I). Then
νF(t) ≤ −1 for t ≤ c and νF(t) ≥ 0 for t > c. Hence ϕ(D*) = F(c)
= f(D − c₁I) + cd. We let k = k(D − c₁I) and show that k = m. Since c > 0,

qₖ(D − c₁I) ≤ dₖ₊₁ − c < d = q(D − c₁I) ≤ qₖ(D − c₁I)

would be a contradiction unless k = m (and dₖ₊₁ does not exist). Now
it is clear that q(D*) = q(D − c₁I) + c. Letting C(D − c₁I) = (c₁, ⋯, cₘ; r), one has C(D*) = (c₁, ⋯, cₘ, c; r) and f(D*) = f(D − c₁I) + c(c₁ + ⋯ + cₘ) + cr = f(D − c₁I) + cd = ϕ(D*).

Finally let there be a c with 0 ≤ c < d₁, d = q(D − (c + 1)I) + 1, and
d = q(D − c₁I) − 1. Then νF(t) ≤ 0 for t < c and νF(t) ≥ 2 for t ≥ c.
Hence ϕ(D*) = F(c) = f(D − c₁I) + cd. Since q(D − c₁I) − q(D − (c + 1)I)
= 2, the lemma tells us that r(D − c₁I) = 0. As before, k(D − c₁I) = m.
Let C(D − c₁I) = (c₁, ⋯, cₘ; 0). One then calculates that q(D*)
= q(D − c₁I) + c − 1, C(D*) = (c₁ − 1, ⋯, cₘ − 1, c; m − 1), and
f(D*) = f(D − c₁I) + (c₁ + ⋯ + cₘ − 1)c = f(D − c₁I) + dc = ϕ(D*). This
completes the proof.

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